Basic Strategies for Solving Logarithmic and Exponential Equations

Case 1: The variable or variable expression is an **input to a log function**: Simplify the equation and **rewrite** the equation in **exponential** format. Solve the resulting equation and check your answer!

Ex. 1: $\text{Log}(\mathbf{20x - 12}) - \text{Log}(\mathbf{4}) = 2$

$\text{Log}\left(\frac{20x-12}{4}\right) = 2$ (Simplify equation)

$\text{Log}(5x - 3) = 2$ (Simplify equation)

$5x - 3 = 10^2 = 100$ (Rewrite in exponential format)

$5x = 103$

$\mathbf{x = 20.6}$ ✓

Ex. 2: $\ln(\mathbf{2x - 3}) + \ln(\mathbf{x}) = 3$

$\ln(2x^2 - 3x) = 3$ (Simplify equation)

(Now rewrite in exponential format)

$2x^2 - 3x = e^3 = 20.085537$

(Set right side of equation equal to zero.)

$2x^2 - 3x - 20.085537 = 0$

(Use the quadratic formula to solve for x.)

$x = 4.006573$ or -2.506573

Since $\ln(-2.506573)$ is undefined,
x = 4.006573 is the only solution! ✓

Ex. 3 $\text{Log}(\mathbf{2x+1}) - \text{Log}(\mathbf{4x - 6}) = 0$

{This is a special case because we can rewrite the equation as Log(expression 1) = Log(expression 2))

$\text{Log}(2x + 1) = \text{Log}(4x - 6)$

Log functions are one-to-one. Therefore if two outputs of a log function are equal, the input values must be equal. We are **NOT** dividing by Log.

$2x + 1 = 4x - 6$

$7 = 2x$

$\mathbf{x = 3.5}$ ✓

Note: This method is faster because we were able to equate two log functions. The standard method just takes longer. See the reverse side of this page.

Case 2: The variable or variable expression is an **exponent** in the equation: Simplify the equation and **take the log of both sides** of the equation. Solve the resulting equation and check your answer!

Ex. 4: $4e^{\mathbf{2x-1}} + 2 = 5$

$e^{2x-1} = ¾ = 0.75$ (Simplify equation)

$2x - 1 = \ln(0.75)$ (Take ln( ) of both sides)

$x = \frac{\ln(0.75) + 1}{2}$

$\mathbf{x = 0.35615896}$ ✓

Ex. 5: $5{,}000(1.075)^{\mathbf{t}} = 20{,}000$

$1.075^t = 4$ (Simplify equation)

$\mathbf{t}\text{Log}(1.075) = \text{Log}(4)$ (Take Log( ) of both sides)

$\mathbf{t} = \frac{\text{Log}(4)}{\text{Log}(1.075)}$

$\mathbf{t = 19.1687}$ ✓

Note: Logs are NOT canceled! !!!! Divide Log(4) by Log(1.075).

Ex. 6: $4^{\mathbf{x+3}} = 8^{\mathbf{2x}}$

This equation is a special case because each side of the equation can be expressed as a power of the same base.

$(2^2)^{x+3} = (2^3)^{2x}$

$2^{2x+6} = 2^{6x}$

(Exponential functions are one-to-one.)

$2x + 6 = 6x$

$6 = 4x$

$\mathbf{x = 1.5}$ ✓

Note: Compare this solution to to the solution shown on the reverse side of this page!

Case 3: The variable or variable expression is a **base** value in the equation: Simplify the equation and raise both sides of the equation to the **reciprocal power** of the base expression. Solve the resulting equation and check your answer!

Ex. 7: $5{,}000(\mathbf{1 + r/12})^{240} = 25{,}000$

$(1 + r/12)^{240} = 5$ (Simplify equation)

$((1 + r/12)^{240})^{1/240} = 5^{1/240}$

$1 + r/12 = 1.0067285$

$r/12 = 0.0067285$

$\mathbf{r = .080742}$ ✓

Ex. 8: $4(\mathbf{x - 1})^3 - 50 = 0$

$(x - 1)^3 = 12.5$ (Simplify equation)

$((x - 1)^3)^{1/3} = 12.5^{1/3}$

$x - 1 = 2.320794417$

$\mathbf{x = 3.320794417}$ ✓

Ex. 9: $5\sqrt[4]{\mathbf{2x + 5}} = 15$

$(2x + 5)^{1/4} = 3$ (Simplify and rewrite equation.)

$((2x + 5)^{1/4})^4 = 3^4$

$2x + 5 = 81$

$2x = 76$

$\mathbf{x = 38}$ ✓

Another method of solving the example **3** equation on the reverse side.
Make sure that you understand both methods.

Log(2x+1) – Log(4x - 6) = 0

$$Log\left(\frac{2x+1}{4x-6}\right)=0$$

Used properties of Log function to rewrite the left side of the equation as the log of a single quantity..

$$\frac{2x+1}{4x-6}=10^0=1$$

Rewrite equation in exponential form.

2x + 1 = 4x - 6

7 = 2x

Use standard equation solving techniques to solve the resulting equation.

x = 7/2 = 3.5 ✔

Another method of solving the example **6** equation on the reverse side of this handout.
Make sure that you understand both methods.

$$4^{x+3}=8^{2x}$$

$$\ln(4^{x+3}) = \ln(8^{2x})$$

Take the natural log of both sides of the equation. We did NOT multiply both sides of the equation by the ln( ) function.

$$(x + 3)\ln(4) = 2x\ln(8)$$

Use properties of ln( ) function to rewrite both sides of the equation.

$$\ln(4)x+3\ln(4)=2x\ln(8)$$

Use distributive property to distribute ln(4) on left side of equation.

$$\ln(4)x - 2x\ln(8) = -3\ln(4)$$

Gather **x** variable on the left side of the equation and constant on the right side.

$$x(\ln(4) - 2\ln(8)) = -3\ln(4)$$

Factor out **x** on left side of equation.

$$x = \frac{-3\ln(4)}{\ln(4) - 2\ln(8)} = \frac{3\ln(4)}{2\ln(8) - \ln(4)}$$

Divide both sides by (ln(4) – 2ln(8) and then multiply numerator and denominator on the right side by -1.

$$x = \frac{\ln(64)}{\ln(64) - \ln(4)} = \frac{\ln(64)}{\ln(16)}$$

Use properties of logs to rewrite 3ln(4) and 2ln(8) and then use properties of ln( ) to simplify the denominator.

x = 1.5 ✔

Use calculator to compute decimal solution.

Special Cases – Rewrite the Equation in Quadratic Format

Example 1: Solve $3e^{x} - 2e^{-x} = 10$

$$e^{x}(3e^{x} - 2e^{-x}) = 10e^{x} \quad \text{(Remove negative exponent.)}$$

$$3e^{2x} - 2e^{0} = 10e^{x}$$

$$3(e^{x})^{2} - 10e^{x} - 2 = 0 \quad \text{(Recognize quadratic format.)}$$

$$e^{x} = \frac{10 \pm \sqrt{124}}{6}$$

$$e^{x} = 3.522588121 \text{ or } -0.1892547876$$

$$\therefore \; e^{x} = 3.522588121 \text{ only! Why?}$$

$$\ln(e^{x}) = x = \ln(3.522588121) = 1.259195981$$

Comment: There are two methods of checking your solutions graphically.

Method 1: Graph the equation $y = 3e^{x} - 2e^{-x} - 10$ and then find the x-intercept of the graph.

Method 2: Graph the equation $y = 3e^{x} - 2e^{-x}$ and the equation $y = 10$. Then find the intersection point of the two graphs.

Example 2: Solve $\ln^{2}(x) - 4\ln(x^{2}) = 1$

$$\ln^{2}(x) - 8\ln(x) - 1 = 0 \quad \text{(Recognize quadratic format.)}$$

$$\ln(x) = \frac{8 \pm \sqrt{64+4}}{2} = \frac{8 \pm \sqrt{68}}{2}$$

$$\ln(x) = 8.123105626 \text{ or } -0.1231056256$$

$$x = e^{8.123105626} \text{ or } x = e^{-0.1231056256}$$

$$\therefore \; x \approx 3{,}371.475044 \text{ or } x \simeq 0.8841702666$$

Comment: You can check your solutions by plugging the solutions into the original equation.

Solution 1: Clear the screen and enter 3371.475044. Then enter the following expression: $\ln(ans)^{2} - 4\ln(ans^{2})$.

Solution 2: Clear the screen and enter 0.8841702666. Then enter the following expression: $\ln(ans)^{2} - 4\ln(ans^{2})$.