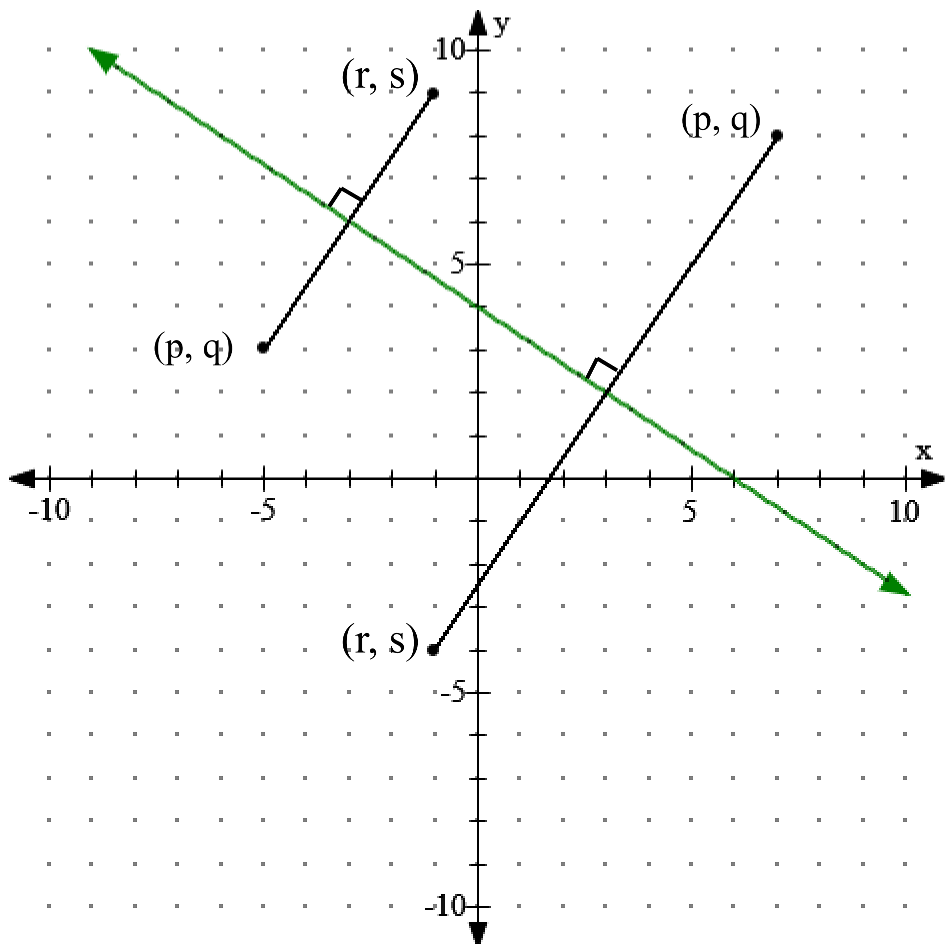
Linear Transformation Rule to Reflect a Figure Over the Line y = mx + b

Problem 1: Find a linear transformation rule of the form (p, q) → (r, s) such that the reflection image of the point (p, q) over the oblique line y = mx + b is the point (r, s). In the general case, both r and s are functions of p, q, m and b. When the values of m and b are known constants, both r and s can be treated as functions of p and q. The slope m of a vertical line is undefined, and the reflection image of the point (p, q) over the vertical line x = k is (-(p – k)+ k, q) = (-p + 2k, q). For the horizontal line y = k, the linear transformation rule is (p, q) → (p, -q + 2k).

Solution: Points (p, q) and (r, s) are reflection images of each other if and only if the line of reflection is the perpendicular bisector of the line segment with endpoints at (p, q) and (r, s). (In the graph below, the equation of the line of reflection is y = -2/3x + 4. Note that both segments have slopes = 3/2, and the shorter segments on both sides of the line of reflection also have slopes = 3/2. If you are using a x-y coordinate axes drawn with a 1:1 aspect ratio, you can find preimage and image points by just counting a certain number of spaces left/right and up/down from any fixed point on the reflecting line; neither a protractor nor ruler is required.)



(Solution continues on next page.)

The segment with endpoints at (p, q) and (r, s) has slope = -1/m since the product of the slopes of oblique perpendicular lines equals -1. Hence the following relationship :

$$(q - s) / (p - r) = -1/m \implies m(q - s) = r - p \implies \boxed{\mathbf{r + ms = p + mq}}$$

The midpoint of the segment at ((p + r)/2, (q + s)/2) necessarily satisfies the equation y = mx + b. Therefore it follows that

$$\begin{aligned}(q + s)/2 = \mathbf{m}(p + r)/2 + \mathbf{b} &\implies q + s = m(p + r) + 2b\\ &\implies q + s = mp + mr + 2b\\ &\implies \boxed{\mathbf{-mr + s = mp - q + 2b}}\end{aligned}$$

Now use the two boxed equations above to setup and solve a system of linear equations for **r** and **s** where **p**, **q**, **m** and **b** are treated as constants.

$$\begin{array}{lllll}(1) & r + ms = p + mq & \xrightarrow{*\ m} & mr + m^2s = & mp + m^2q\\ (2) & -mr + s = mp - q + 2b & \longrightarrow & -mr + s = & mp - q + 2b\end{array}$$

Hence $s(m^2 + 1) = 2mp + m^2q - q + 2b \implies \boxed{\mathbf{s = (m^2q + 2mp - q + 2b) / (m^2 + 1)}}$

Equation (1) implies $r = p + mq - m\mathbf{s}$

$$\begin{aligned} r &= p + mq - m(m^2q + 2mp - q + 2b) / (m^2 + 1)\\ r &= p + mq + (-m^3q - 2m^2p + mq - 2bm) / (m^2 + 1)\\ r &= (pm^2 + p + m^3q + mq - m^3q - 2m^2p + mq - 2bm) / (m^2 + 1)\\ &\boxed{\mathbf{r = (-m^2p + 2mq + p - 2bm) / (m^2 + 1)}}\end{aligned}$$

Rewriting the two boxed equations for r and s, we have (p, q) $\rightarrow$ (r, s) where

$$\boxed{\begin{aligned} r &= \left(\frac{1 - m^2}{m^2+1}\right)p + \left(\frac{2m}{m^2+1}\right)q - \left(\frac{2mb}{m^2+1}\right)\\ s &= \left(\frac{2m}{m^2+1}\right)p + \left(\frac{m^2-1}{m^2+1}\right)q + \left(\frac{2b}{m^2+1}\right)\end{aligned}}$$

The transformation (p, q) $\rightarrow$ (r, s) expressed in terms of matrix multiplication is given below.

$$\boxed{\begin{bmatrix} r\\ s\\ 1\end{bmatrix} = \frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m & -2bm\\ 2m & m^2-1 & 2b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} p\\ q\\ 1\end{bmatrix}}$$

Problem 2: Let m = the slope of an oblique line, angle $\theta = Tan^{-1}(m)$, and (0, b) equal the y-intercept of the line. Use the results from problem 1 to find a geometric transformation rule of the form $(p, q) \rightarrow (r, s)$ such that the reflection image of the point (p, q) over the line y = mx + b is the point (r, s) where both r and s are functions θ and b.

Solution: Using basic trig identities and the fact that m = Tan(θ), we have the following results:

1. $m^2+1 = \mathrm{Tan}^2(\theta) + 1 = \mathrm{Sec}^2(\theta)$
2. $1 - m^2 = \dfrac{Cos^2(\theta)-Sin^2(\theta)}{Cos^2(\theta)} = \dfrac{Cos(2\theta)}{Cos^2(\theta)}$
3. $m^2 - 1 = -\dfrac{Cos(2\theta)}{Cos^2(\theta)}$
4. $\dfrac{1-m^2}{1+m^2} = \dfrac{\dfrac{Cos(2\theta)}{Cos^2(\theta)}}{Sec^2(\theta)} = \dfrac{Cos(2\theta)}{Cos^2(\theta)\mathrm{Sec}^2(\theta)} = Cos(2\theta)$
5. $\dfrac{m^2-1}{1+m^2} = -Cos(2\theta)$
6. $\dfrac{2m}{1+m^2} = \dfrac{\dfrac{2\mathrm{Sin}(\theta)}{Cos(\theta)}}{\dfrac{Cos^2(\theta)+Sin^2(\theta)}{Cos^2(\theta)}} = 2\mathrm{Sin}(\theta)\mathrm{Cos}(\theta) = \mathrm{Sin}(2\theta)$
7. $\dfrac{2}{1+m^2} = \dfrac{2}{Sec^2(\theta)} = 2\mathrm{Cos}^2(\theta)$

Using the results from problem 1 and the relationships stated in equations 1 through 7 above, it follows that $(p, q) \rightarrow (r, s)$ where

$$\boxed{\begin{aligned} r &= \mathrm{Cos}(2\theta)p + \mathrm{Sin}(2\theta)q - \mathrm{Sin}(2\theta)b \\ s &= \mathrm{Sin}(2\theta)p - \mathrm{Cos}(2\theta)q + 2\mathrm{Cos}^2(\theta)b \end{aligned}}$$

The transformation $(p, q) \rightarrow (r, s)$ expressed in terms of matrix multiplication is shown below.

$$\boxed{\begin{bmatrix} r \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} Cos(2\theta) & Sin(2\theta) & -Sin(2\theta)b \\ Sin(2\theta) & -Cos(2\theta) & 2Cos^2(\theta)b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}}$$

Examples:

1. If the line of reflection is the x-axis, then $m = 0$, $b = 0$, and $(p, q) \longrightarrow (p, -q)$.

2. If the line of reflection is $y = x$, then $m = 1$, $b = 0$, and $(p, q) \longrightarrow (2q/2, 2p/2 = (q, p)$.

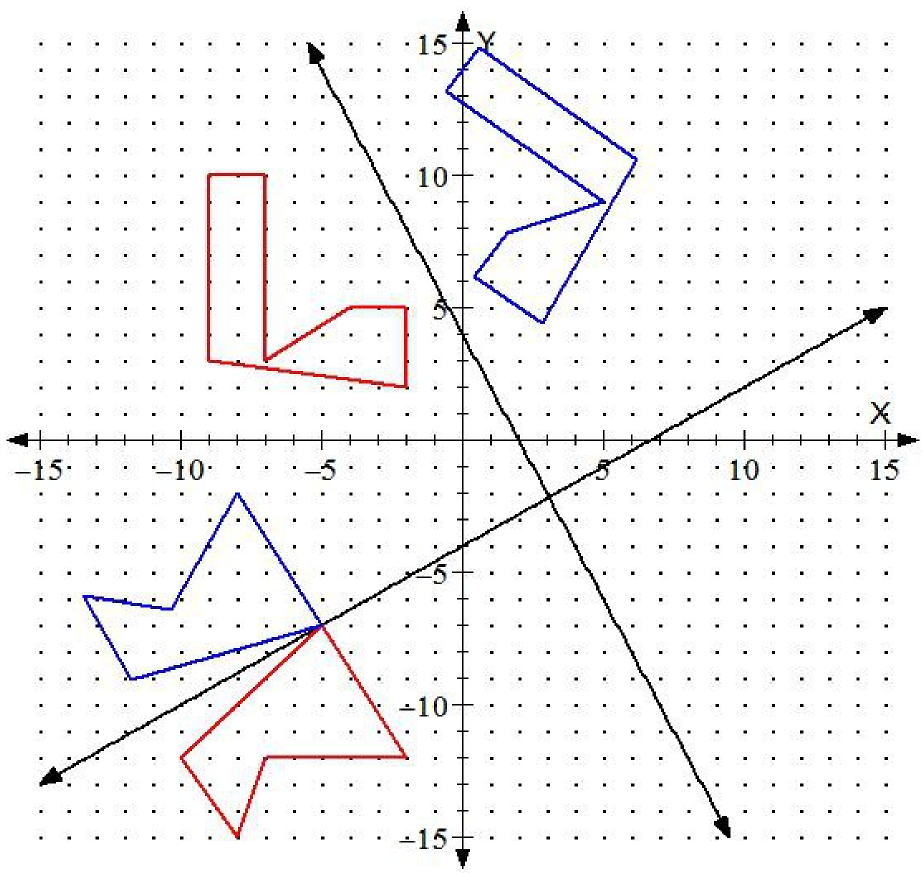
3. If the line of reflection is $y = -2x + 4$, then $m = -2$, $b = 4$, $(1 - m^2)/(1 + m^2) = -3/5$, $(m^2 - 1)/(m^2 + 1) = 3/5$, $2m/(m^2 + 1) = -4/5$, $-2mb/(m^2 + 1) = 16/5$, and $2b/(m^2 + 1) = 8/5$.

   $(p, q) \longrightarrow (-3/5p - 4/5q + 16/5 , - 4/5p + 3/5q + 8/5)$

4. If the line of reflection is $y = 3/5x - 4$, then $m = 3/5$, $b = -4$, $(1 - m^2)/(1 + m^2) = 8/17$, $(m^2 - 1)/(m^2 + 1) = -8/17$, $2m/(m^2 + 1) = 15/17$, $-2mb/(m^2 + 1) = 60/17$, and $2b/(m^2 + 1) = -100/17$.

   $(p, q) \longrightarrow (8/17p + 15/17q + 60/17, 15/17p - 8/17q - 100/17)$

The graph below shows the reflection images of two polygons over the lines $y = -2x + 4$ and $y = 3/5x - 4$. Refer to examples (3) and (4) above for the derivation of the linear transformation rules.



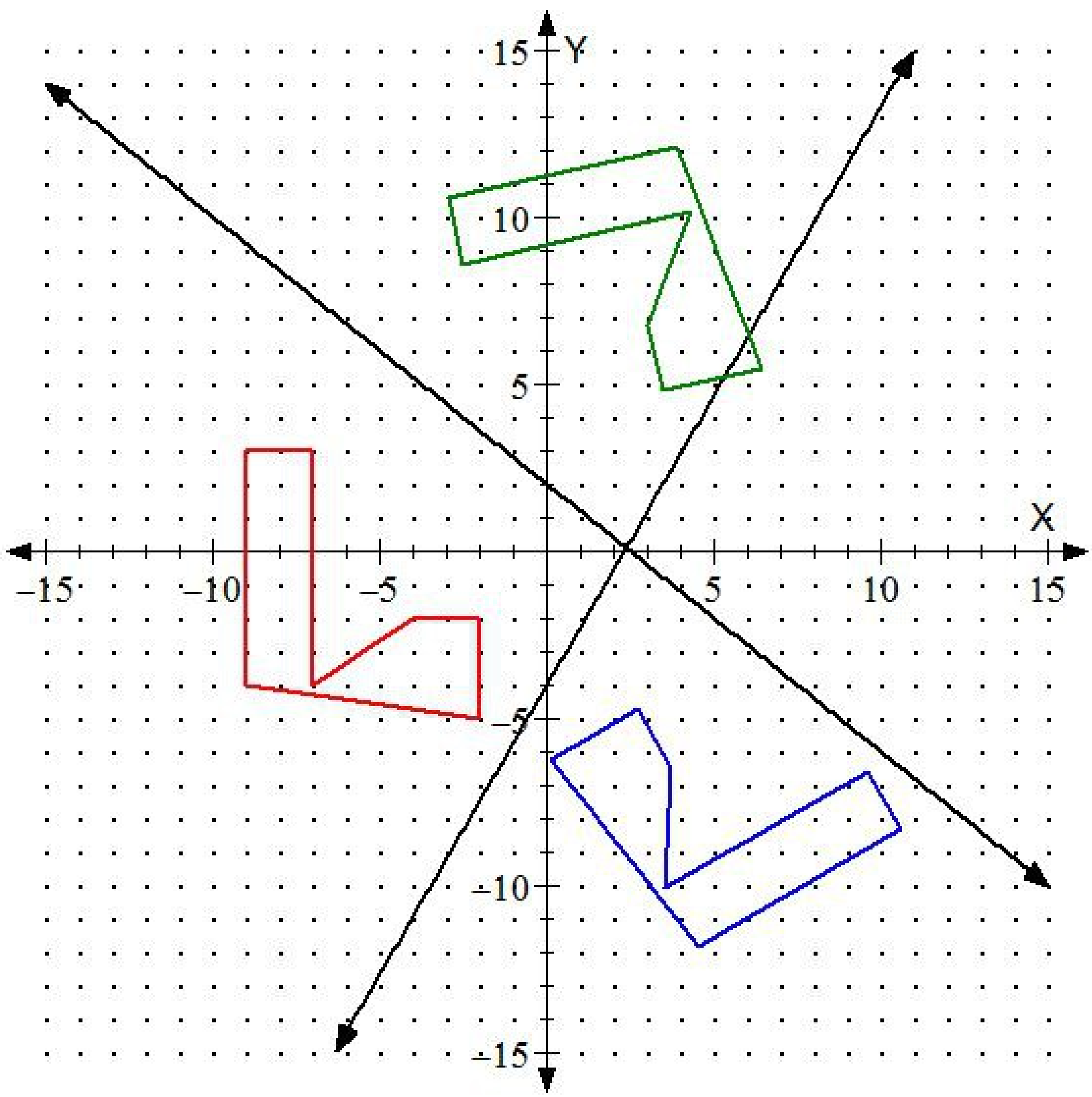
5. If the line of reflection is $y = \sqrt{(3)}x - 4$, then $m = \sqrt{(3)}$, $\theta = Tan^{-1}(\sqrt{(3)}) = 60^0$ and $b = -4$. $Sin(2\theta) = 0.866025$, $Cos(2\theta) = -0.5$, $-bSin(2\theta) = 3.464102$, and $2bCos^2(\theta) = -2$.

   $(p, q) \longrightarrow (-0.5p + 0.866025q + 3.464102, 0.866025p + 0.5q - 2)$

6. If the line of reflection is $y = -4/5x + 2$, then $m = -4/5$, $\theta = Tan^{-1}(-4/5) = -38.659808^0$ and $b = 2$. $Sin(2\theta) = -0.975610$, $Cos(2\theta) = 0.219512$, $-bSin(2\theta) = 1.9512220$, and $2bCos^2(\theta) = 2.439024$.

   $(p, q) \longrightarrow (0.219512p - 0.975610q + 1.9512220, -0.976510p - 0.219512q + 2.439024)$

The graph below shows the reflection images of a polygons over the lines $y = \sqrt{(3)}x - 4$ and $y = -4/5x + 2$. Refer to examples (5) and (6) above for the derivation of the linear transformation rules.



Algorithm to Draw Preimage and Image Polygons Under a Linear Geometric Transformation with the TI-84 Graphing Calculator

1) Press **MODE** button to set standard x-y coordinate graphing parameters. Press **QUIT** button when finished.

2) Use **STAT/EDIT/ClrList** command to clear lists L_1, L_2, L_3, L_4, L_5, L_6

3) Determine the values of the six constants c_1, c_2, c_3, c_4, c_5, c_6 that define the linear transformation $(p, q) \longrightarrow (r, s)$ where $r = c_1p + c_2q + c_3$ and $s = c_4p + c_5q + c_6$.

4) Determine the x-y coordinates of the **n** vertices of the preimage polygon (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n).

5) Create a written list of **n+1** x-y coordinates of the vertices of the preimage polygon by setting the last point in the list to the first vertex of the polygon. In other words, repeat the first vertex of the preimage polygon.

6) Use **STAT/EDIT** command to enter the **n+1** x-coordinates in L_1 and the **n+1** y-coordinates in L_2. Press **QUIT** when finished.

7) The **STO** button causes the store symbol $\longrightarrow$ to be pasted in the calculator's main window.
   a) Enter the transformed x-coordinates in L_3 by typing the command $c_1 L_1 + c_2L_2 + c_3 \longrightarrow L_3$
      Example: $0.6L_1 – 1.75L_2 + 2.3456 \longrightarrow L_3$
   b) Enter the transformed y-coordinates in L_4 by typing the command $c_4L_1 + c_5L_2 + c_6 \longrightarrow L_4$

8) If you wish to view the x-y coordinates of the vertices of the image polygon in L_3 and L_4, use the **STAT/EDIT** command. Press **QUIT** button when finished viewing.

9) Press **WINDOW** button to set appropriate values for Xmin, Xmax, Xsch, Ymin, Ymax and Yscl

10) Press **STAT PLOT** button to tell the graphing calculator which graphs to draw.
   a) Set **PLOT 1** to **On**. Select **Polygon icon** in middle of top row. Xlist: L_1 and Ylist: L_2
   b) Set **PLOT 2** to **On**. Select **Polygon icon** in middle of top row. Xlist: L_3 and Ylist: L_4
   c) Set **PLOT 3** to **Off**

11) Press **Y =** button and enter the equations of any lines you wish to graph.

12) Press **GRAPH** button to view the graphs of the preimage and image polygons.

13) Press **TRACE** button and arrow keys to view x-y coordinates of points on a graph. Use **Up** and **Down** arrow buttons to switch from one graph to another graph. Use **Left** and **Right** arrow buttons to move from one point on a graph to another point on the same graph.

Note: Because the graphs are not drawn with a 1:1 aspect ratio, screen images will be distorted.