Rapid Curve Sketching

In many practical situations it is not necessary to have a super accurate graph of a relation. By making a few simple observations and calculations, one can quickly produce a sketch of the graph of a relation which will be sufficient for most practical applications.

1. Oblique line: $y = mx + b$ **or** $Ax + By = C$

First plot the y-intercept (0,**b**). From the point (0,**b**), plot additional points by moving up/down and left/right in accordance with the value of the slope **m** of the line. If the equation of the line is in standard form, $Ax + By = C$, compute the **x** and **y**-intercepts. Substitute zero for **x** and solve the resulting equation for **y**. Then substitute zero for **y** and solve the resulting equation for **x**.

2. Parabola: $\mathbf{y = ax^2 + bx + c}$ (Expanded or polynomial form)

a) If $a > 0$, parabola opens upward.
If $a < 0$, parabola opens downward.

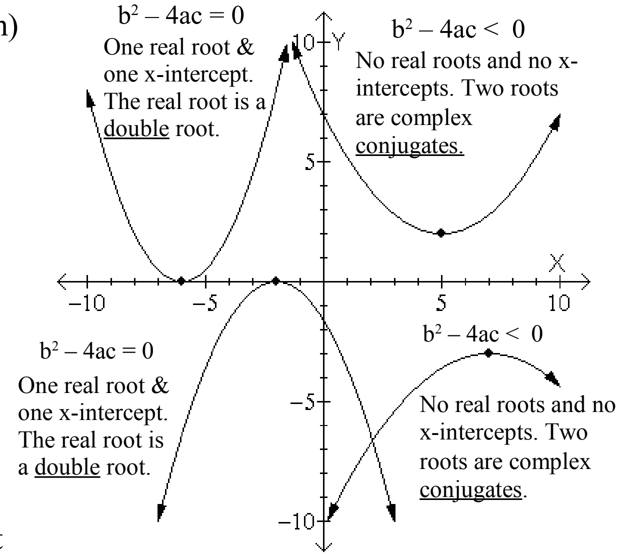
b) y-intercept = (0,**c**)

c) Roots at $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac < 0$, the parabola has **no** x-intercepts. Parabola is entirely above or below x-axis. Roots are complex numbers.

If $b^2 - 4ac = 0$, the parabola has **exactly one** x-intercept and **one** real root. Vertex of parabola kisses the x-axis.

If $b^2 - 4ac > 0$, the parabola has **exactly two** x-intercepts and **two** real roots.



Note: Use the completing the square procedure to convert from $y = ax^2 + bx + c$ format to $y = k(x \pm a)^2 \pm b$ format. See part (3) below.

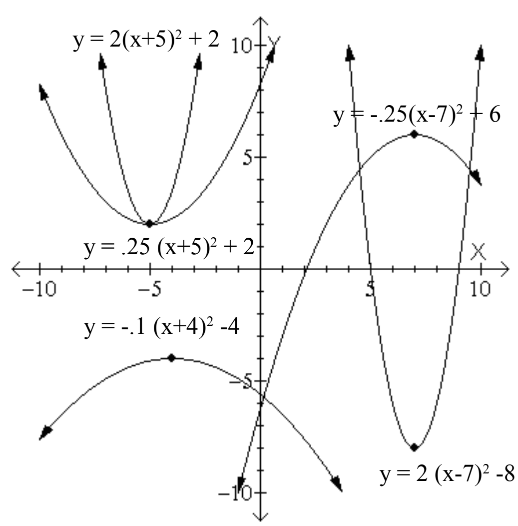
d) **x**-coordinate of the vertex $= x_{min}$ or $x_{max} = \dfrac{\mathbf{-b}}{\mathbf{2a}}$ = the **average** of the real or complex roots.

e) Find y-coordinate of vertex: $y_{min} = f(x_{min})$ and $y_{max} = f(x_{max})$

3. Parabola: $\mathbf{y = k(x \pm a)^2 \pm b}$ (Vertex form of equation)
(Assume a and b are positive)

a) If the absolute value of k increases, parabola gets steeper.
b) If the absolute value of k decreases, parabola gets wider.
c) If $k > 0$, parabola opens upward.
d) If $k < 0$, parabola opens downward.

e) If **(x – a)**, parabola slides **a** units to the right. **(+)**
f) If **(x + a)**, parabola slides **a** units to the left. **(-)**
g) If **+ b**, parabola is shifted vertically upward. **(+)**
h) If **- b**, parabola is shifted vertically downward **(-)**



4. Polynomial functions in <u>factored form</u>. If the polynomial is not in factored form, then factor the polynomial

a) First find the leading term (largest power of x) and constant term of the polynomial.

b) Use the leading term of the polynomial to determine the left and right end behavior of the graph. For very large positive x or extreme negative values of x, the other terms of the polynomial have relatively little effect on the end behavior of the graph.

Left side: Mentally determine the sign value of the leading term for extreme negative values of x.
If the value of the leading term is negative, graph starts below x-axis.
If the value of the leading term is positive, graph starts above x-axis.

Right side: Mentally determine the sign value of the leading term for very large positive values of x.
If the value of the leading term is negative, graph finishes below x-axis.
If the value of the leading term is positive, graph finishes above x-axis.

c) The constant term of the polynomial is the y-intercept of the graph.

d) Each **linear** factor of the polynomial determines a zero root of the polynomial or x-intercept of the graph of the polynomial. The **power of the factor** of the polynomial is the **order** of the root.

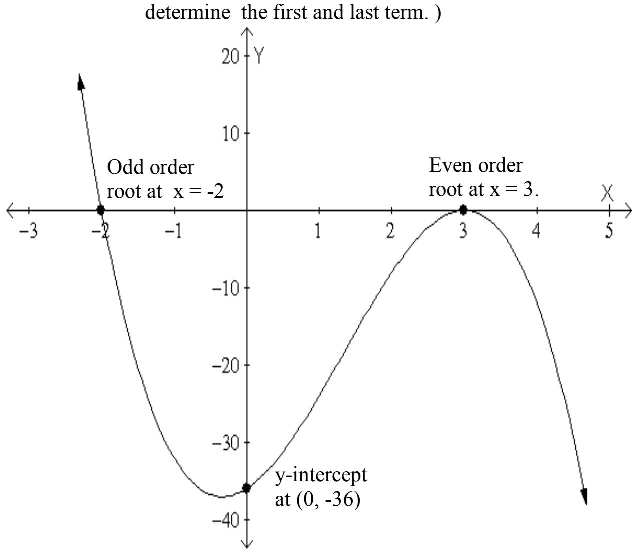
1) If the **power** of the factor is **odd,** we say the **order** of the root is **odd**. At real roots of <u>odd</u> order, the graph of the polynomial <u>cuts through</u> the x-axis at the root.

2) If the **power** of the factor is **even**, we say the **order** of the root is **even**. At real roots of <u>even</u> order, the graph of the polynomial <u>just touches or kisses</u> the x-axis at the root.

Example 1: $y = -2(x + 2)(x - 3)^2$

$y = -2x^3 + \ldots - 36 \implies f(0) = -36$

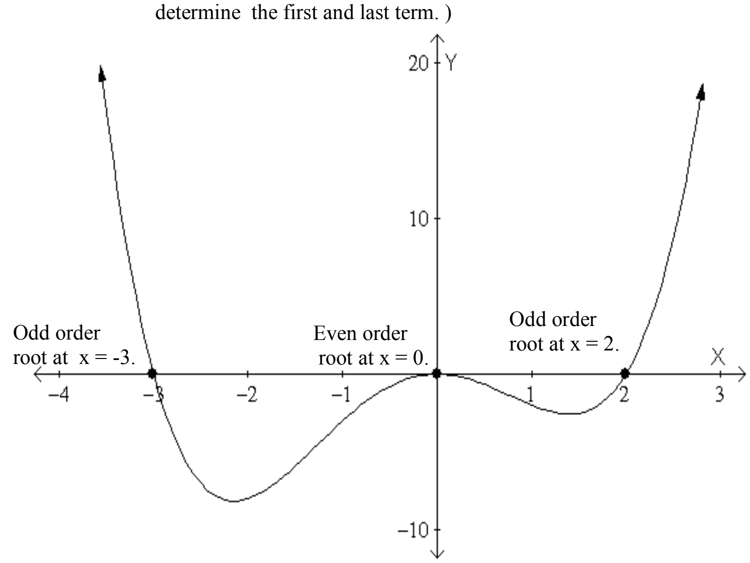
(Mentally expand the polynomial in order to determine the first and last term.)



Example 2: $y = .5x^2(x - 2)(x + 3)$

$y = .5x^4 + \ldots - 3x^2 \implies f(0) = 0$

(Mentally expand the polynomial in order to determine the first and last term.)



5. Rational polynomial functions where both the numerator and denominator are factored polynomials.

a) The real roots of the **numerator** determine the real roots of the rational polynomial function or **x**-intercepts of the graph. The same observations about roots of factored polynomials apply to roots of rational polynomial expressions.

b) The real roots of the **denominator** determine the points where the rational polynomial function is **undefined.** At undefined points, the graph of f(x) climbs up towards positive infinity or drops down towards negative infinity about an invisible vertical line. This invisible line is called a **vertical asymptote** or **pole** of the graph. The **power** of a factor in the denominator determines the **order** of the pole.

At **odd order** poles the graph of f(x) will approach minus infinity on one side of the pole and positive infinity on the other side of the pole. The graph **changes sign** as you cross the pole left to right.

At **even order** poles the graph will approach positive infinity on **both** sides of the pole or negative infinity on **both** sides of the pole. The graph **does not change sign** as you cross the pole left to right.

c) The ratio of the leading term of the numerator to the leading term of the denominator determines the behavior of the graph as **x** approaches minus infinity and positive infinity.

i) If the ratio of the leading term of the numerator and to the leading term of the denominator is a constant **k**, then the graph has a horizontal asymptote of y = **k** . See examples **1 and 4** below.

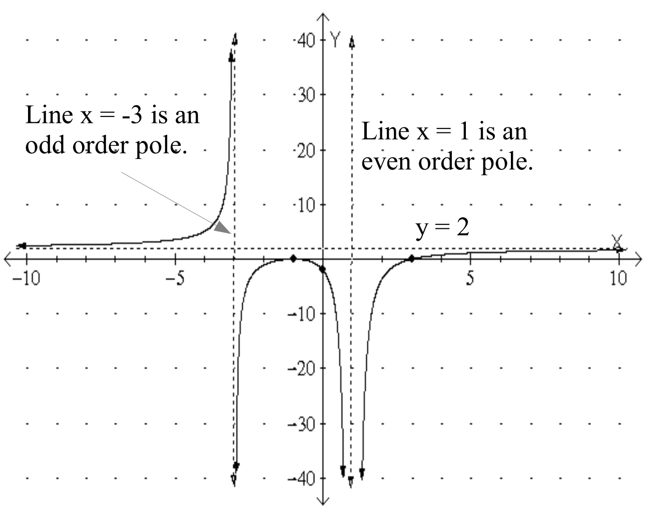
ii) If the degree of the numerator is greater than the degree of the denominator, then f(x) will approach positive infinity **or** negative infinity for large absolute values of **x**. (See example **2**).

iii) If the degree of the numerator is less than the degree of the denominator, then f(x) will approach zero for large absolute values of **x**. The x-axis is a horizontal asymptote. (See example **3** below.)

Example 1: $y = \dfrac{2(x+1)^2\,(x-3)}{(x-1)^2\,(x+3)}$ (Mentally expand the numerator & denominator.)

$y = \dfrac{2x^3 + \ldots \quad - 6}{x^3 + \ldots \quad + 3} \approx 2$ (When absolute value of x is **very** large.)

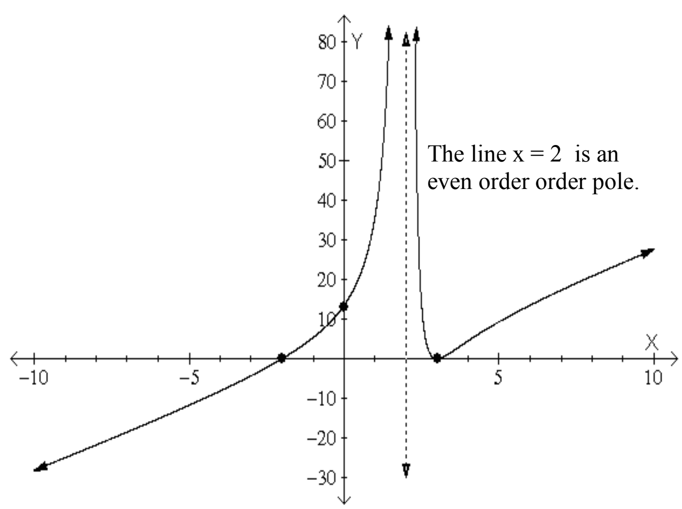
f(0) = -2 (**Why?**)



Example 2: $y = \dfrac{3(x+2)(x-3)^2}{(x-2)^2}$ (Mentally expand the numerator & denominator.)

$y = \dfrac{3x^3 \ldots + 54}{x^2 - 4x + 4} \approx 3x$ (When absolute value of x is **very** large.)

f(0) = 13.5 (**Why?**)



Example 3: $f(x) = \dfrac{6(x^2 + 1)}{(x - 1)^2(x + 2)}$

Analysis:

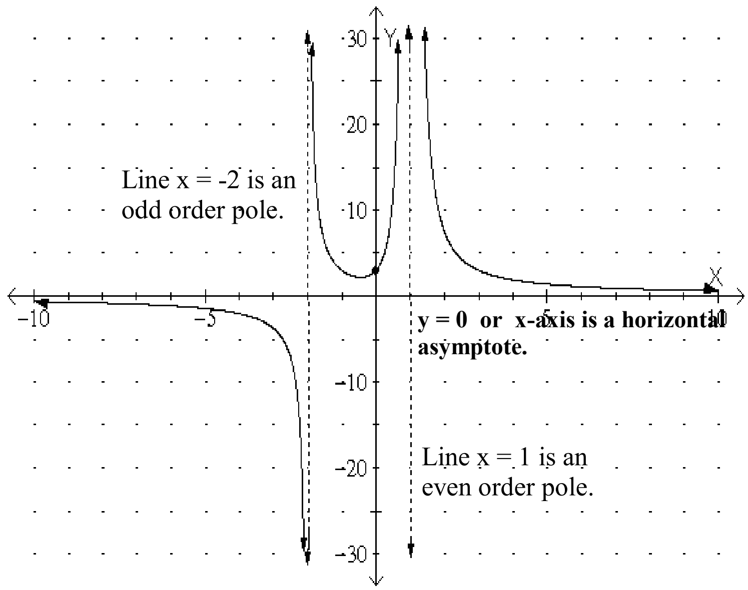
$f(x) = \dfrac{6x^2 + 6}{x^3 + \ldots + 2} \approx \dfrac{6}{x}$ (When absolute value of x is **very** large.)

f(0) = 3 ==> y-intercept at **(0,3)**

Since **$x^2 + 1$** can never equal zero, (**Why?**) f(x) can never equal zero. Hence the graph of f(x) can not intersect the x-axis.

Since f(x) ≈ 6/x when x is very large positive, f(x) is close to zero and positive. x-axis is a horizontal asymptote of the graph.

Since f(x) = 6/x when x is extremely negative, f(x) is close to zero and negative. x-axis is a horizontal asymptote of the graph.



Roots of f(x)	**Poles of graph of f(x)**
None	x = 1 is an even order pole.
	x = -2 is an odd order pole.

Check: f(-10,000) = **0.000 599 760**

f(-2.001) = ______________________

f(-1.999) = ______________________

f(1.001) = ______________________

f(10,000) = ______________________

Example 4: $f(x) = \dfrac{-2x^2 + 18}{x^2 + 12} \approx$ **-2** (When absolute value of x is **very** large.)

How could you tell that the graph is symmetric with respect to the y-axis by just inspecting the equation of f(x)?

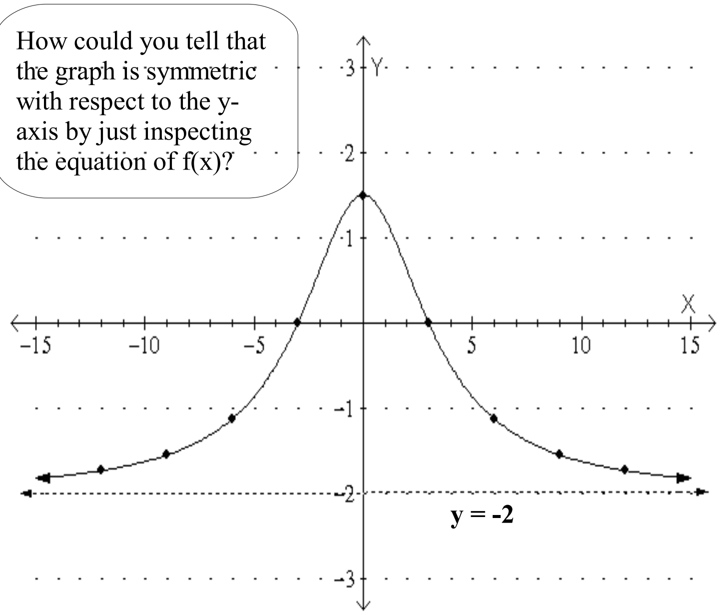
Analysis:

f(0) **=** 18/12 **= 1.5** ===> y-intercept at **(0, 1.5)**

$f(x) = \dfrac{-2(x^2 - 9)}{x^2 + 12} = \dfrac{-2(x + 3)(x - 3)}{x^2 + 12}$

Since $x^2 + 12$ can never equal zero, (**Why?**) the graph of f(x) has **no** poles or vertical asymptotes.

For very large positive **x** and extremely negative **x**, f(x) ≈ -2. Hence the line y = -2 is a horizontal asymptote of the graph of f(x).



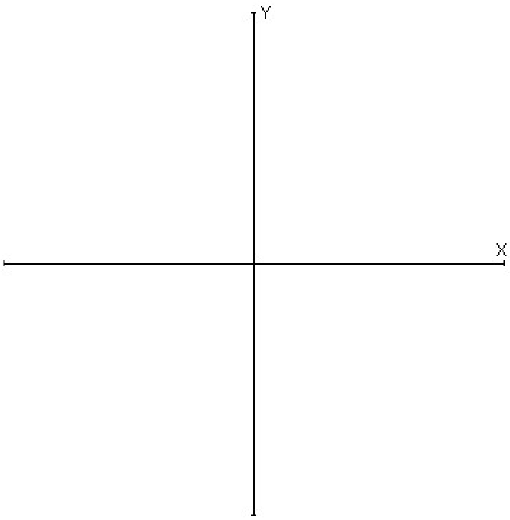
Roots of f(x)	**Poles**
x = -3 is an odd order root	None
x = 3 is an odd order root	

Check: f(10,000) = -1.99999958

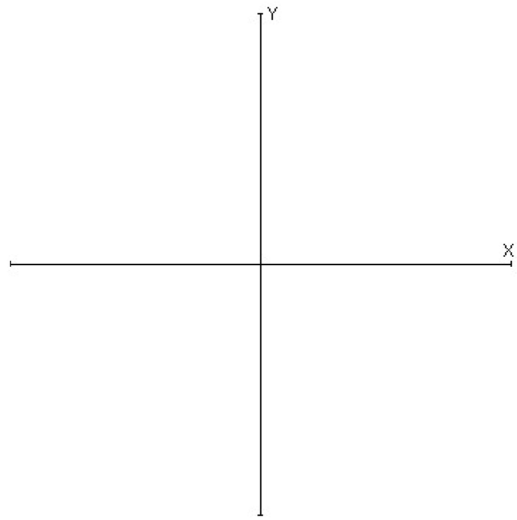
f(-10,000) = -1.99999958

Do a rapid curve sketch of each function. **Name** ________________________________

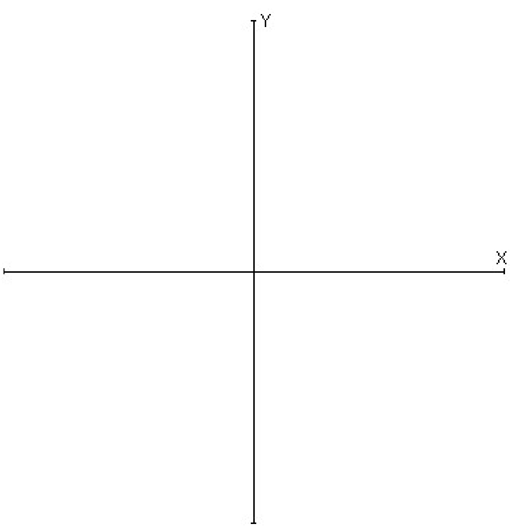
1. $\mathbf{y = -2x^2 - 8x + 10}$



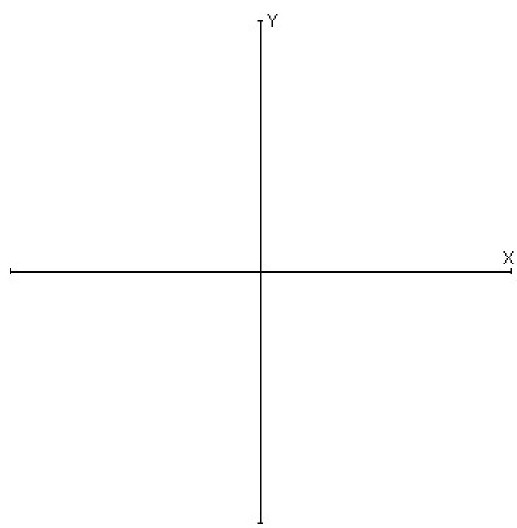
2. $f(x) = (x - 2)(x + 4)^2$



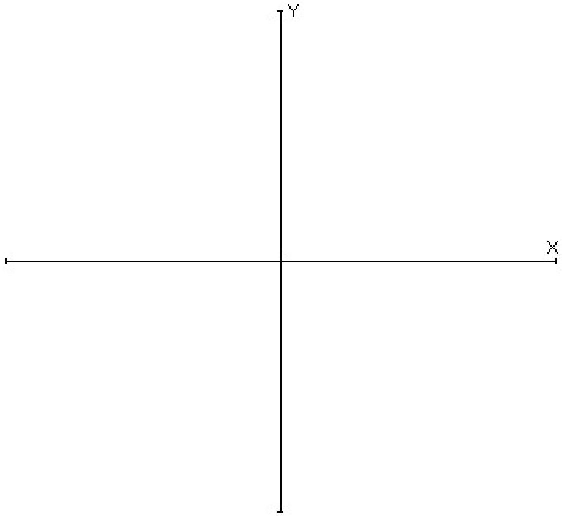
3. $y = 3x^2 - 12x$



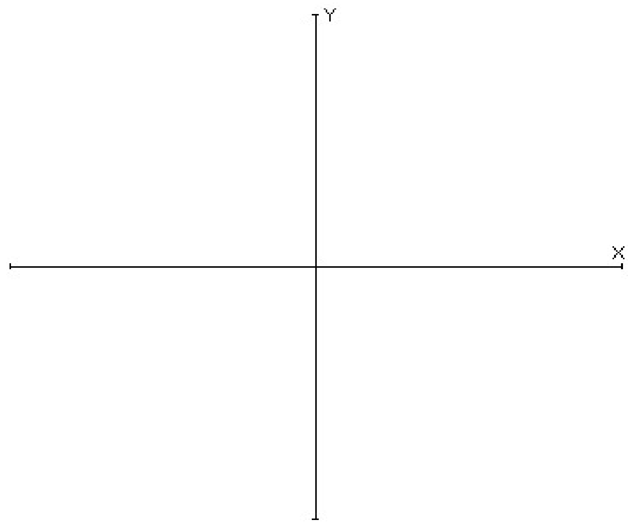
4. $g(x) = (2x + 7)(x - 2)^3(x + 5)^2$



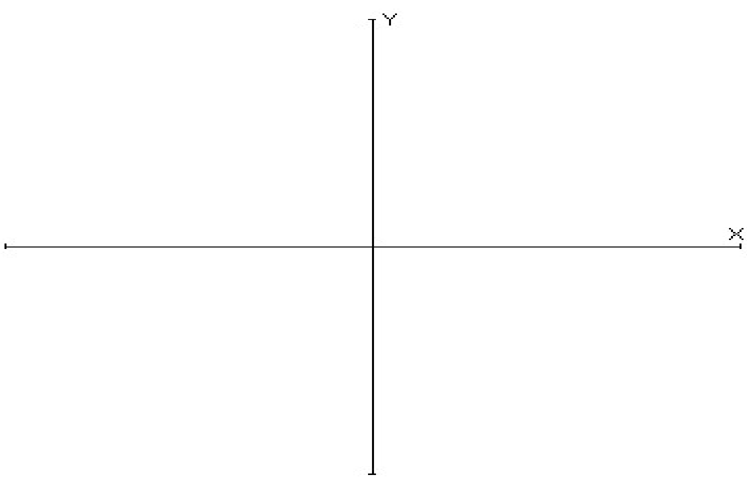
5. $h(x) = 2x^2 + 6x + 10$



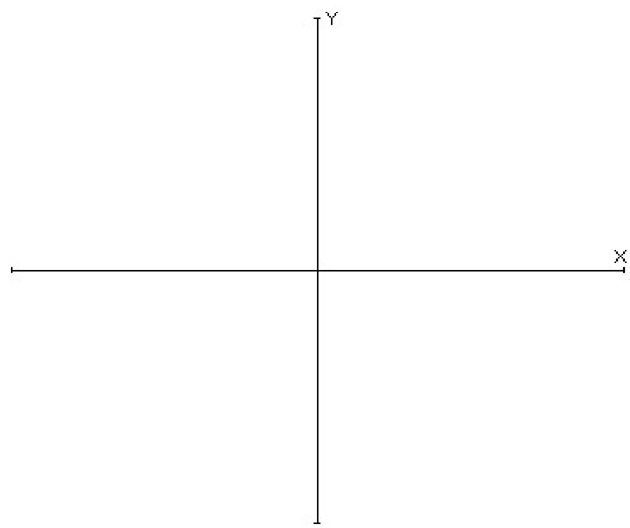
6. $y = \dfrac{2x(x + 5)}{(x - 3)(x + 7)}$



7. $y = \dfrac{(2x - 6)\,(x + 4)^2}{(x + 1)^2\,(x - 5)}$

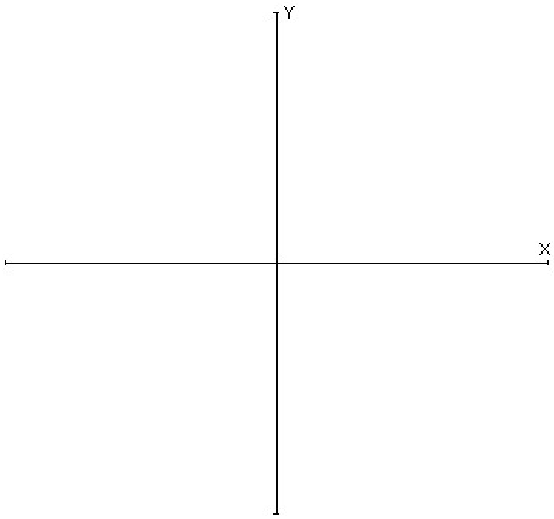


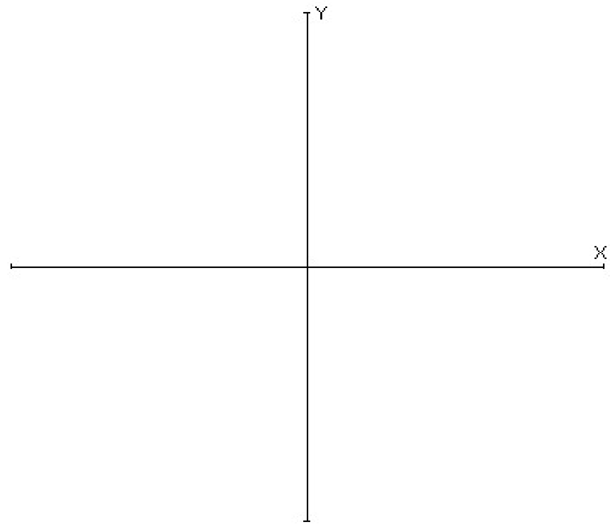
8. $y = -2(x+3)^2(x+2)(x-3)$



9. $f(x) = \dfrac{x^3 + x^2 + 6x}{x^2 - 1}$

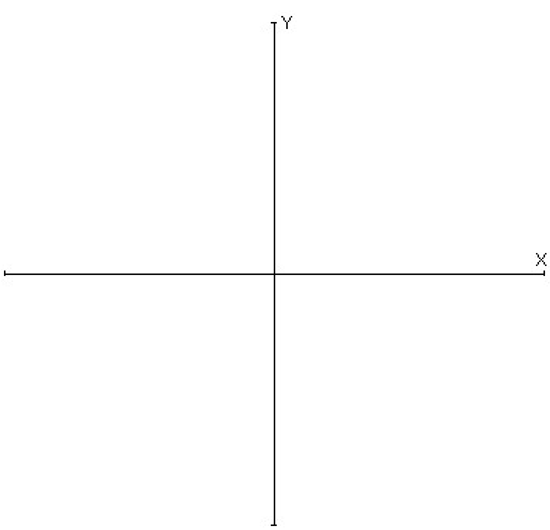
10. $g(x) = \dfrac{x - 1}{x^2(x+2)^2}$

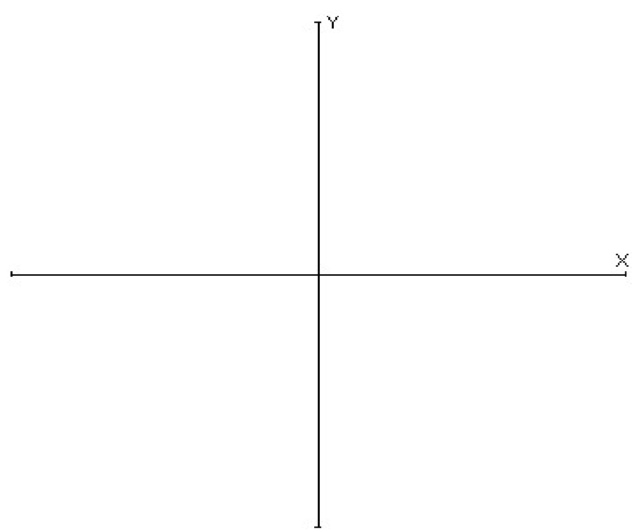




11. $f(x) = \dfrac{4x^2}{x^2 + 5}$

12. $h(x) = \dfrac{4x}{x^2 + 5}$





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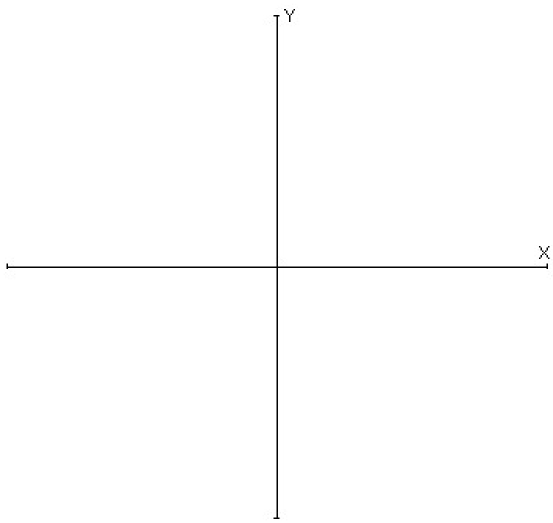

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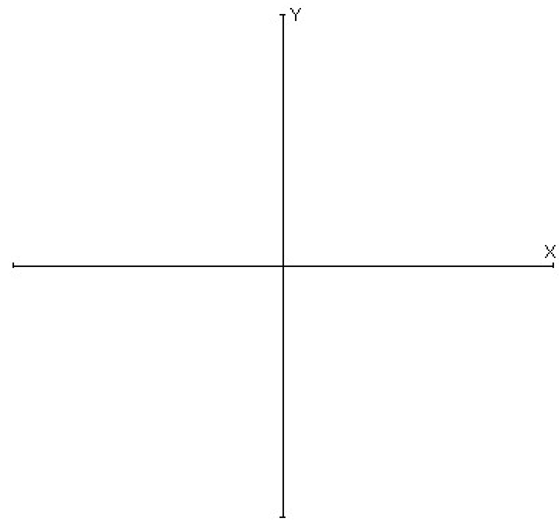

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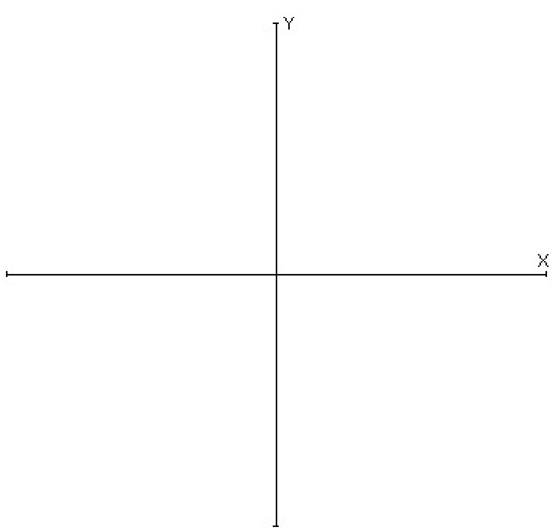

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