Quadratic Formula - Key

Key

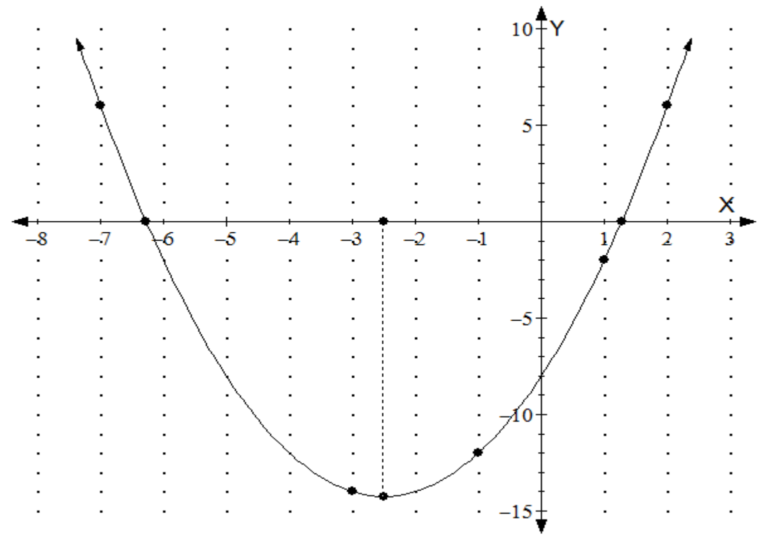
Background: So far in this course we have solved quadratic equations by the square root method and the factoring method. Each of these methods has its strengths and limitations. The quadratic formula allows us to solve <u>any</u> quadratic equation for <u>both</u> the real and complex <u>roots</u>. The derivation of the quadratic formula uses the completing square technique applied to a general quadratic equation of the form $ax^2 + bx + c = 0$.

Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1. The graph of $y = x^2 + 5x - 8$ is shown below.

a) Fill in the table and plot the points on the graph.

x	y = f(x)
-7	6
-3	-14
-1	-12
1	-2
2	6
3	16



a) What is the **y**-intercept of the graph?

(0, -8)

b) What are the **x**-intercepts of the graph or the <u>real roots</u> of f(x)? Round to 4 decimal places.

a = 1, b = 5, and c = -8

$$x = \frac{-5 \pm \sqrt{25 - 4(-8)}}{2}$$

$$x = \frac{-5 \pm \sqrt{57}}{2} \Rightarrow x = -6.2749 \text{ or } 1.2749$$

x-intercepts: (-6.2749, 0) **or** (1.2749, 0)

Roots: x = -6.2749 **or** x = 1.2749

c) What value of **x** does the graph reach its minimum value and what is the minimum value ?

X_{min} = the average of the two roots
= (-6.2749 + 1.2749) / 2 = -2.5.

$Y_{min} = f(x_{min}) = f(-2.5) = -14.25$

$\mathbf{x_{min}}$ = -2.5 $\mathbf{y_{min}}$ = -14.25

d) What values of **x** make **y** = 8 ? (If f(x) = 8, what are the values of x?)

$x^2 + 5x - 8 = -8$ ====> $x^2 + 5x - 16 = 0$
a = 1, b = 5, and c = -16

$$x = \frac{-5 \pm \sqrt{25 - 4(-16)}}{2}$$

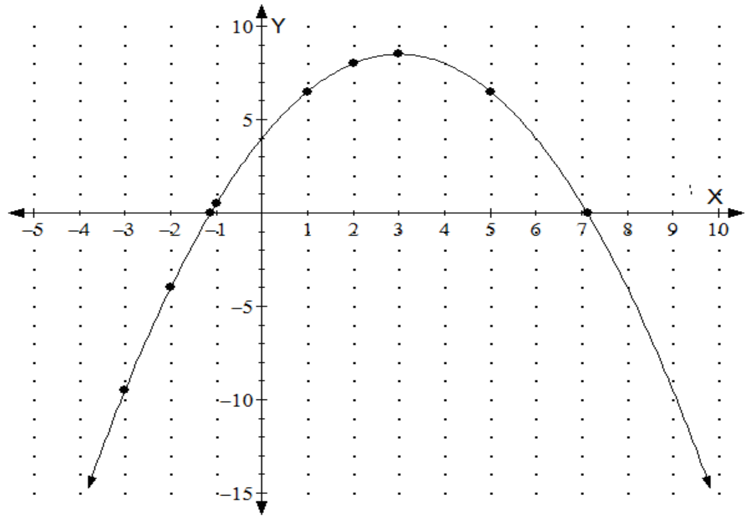
$$x = \frac{-5 \pm \sqrt{89}}{2} \Rightarrow x \approx -7.2170 \text{ or } 2.2170$$

f(x) = 8 when x = -7.2170 or x = 2.2170.

2. The graph of **$y = -0.5x^2 + 3x + 4$** is shown below.

a) Fill in the table and plot the points on the graph.

x	y = f(x)
-3	-9.5
-2	-4
-1	0.5
1	6.5
2	8
5	6.5



b) What is the y-intercept of the graph?

(0, 4)

c) What are the x-intercepts of the graph or the real roots of f(x)? Round to 4 decimal places.

$-0.5x^2 + 3x + 4 = 0$

a = -0.5, b = 3, and c = 4 .

$$x = \frac{-3 \pm \sqrt{9-4(-2)}}{-1} = 3 \pm \sqrt{17}$$

x ≈ -1.1231 or 7.1231

x-intercepts: (-1.1231, 0) **or** (7.1231, 0)

Roots: x = -1.1231 **or** x = 7.1231

d) What value of **x** does the graph reach its maximum value and what is the maximum value ?

X_{max} = the average of the two roots
= (-1.1231 + 7.1231) / 2 = 3.

$Y_{max} = f(x_{max}) = f(3) = 8.5$

x_{max} = 3 y_{max} = 8.5

e) What values of **x** makes **y** = 7 ?

$-0.5x^2 + 3x + 4 = 7$

$-0.5x^2 + 3x - 3 = 0$

a = -0.5, b = 3, and c = -3 .

$$x = \frac{-3 \pm \sqrt{9-4(1.5)}}{-1} = 3 \pm \sqrt{3}$$

x ≈ 1.2679 or x ≈ 4.7321

f(x) = 7 when x = 1.2679 or x = 4.7321

f) What values of **x** make **y** = - 8 .

$-0.5x^2 + 3x + 4 = -8$

$-0.5x^2 + 3x + 12 = 0$

a = -0.5, b = 3, and c = 12 .

$$x = \frac{-3 \pm \sqrt{9-4(-6)}}{-1} = 3 \pm \sqrt{33}$$

x ≈ -2.7446 or x ≈ 8.7446

f(x) = -8 when x = -2.7446 or x = 8.7446

3. The height above ground of a ball thrown upward with an initial vertical velocity of 84 ft/ sec is described by the equation $\mathbf{h = -16t^2 + 84t}$ where **h** = height in feet above ground at time **t** and **t** = time in seconds.

84 ft/sec = 57.3 mph .

a) How high above ground is the ball when t = 1 second ? 68 feet

b) How high above ground is the ball when t = 2 seconds ? 104 feet

c) What was the average vertical speed of the ball over the time interval [1 sec, 2 sec] ? 36 ft/sec

Height of ball increased 36 ft in 1 second.

d) How high above ground is the ball when t = 4 second ? 80 feet

e) How high above ground is the ball when t = 4.5 seconds ? 54 feet

f) What was the average vertical speed of the ball over the time interval [4 sec, 4.5 sec] ? -52 ft/sec

Height of ball decreased 26 ft in 0.5 seconds.

g) At what time **t** did the ball hit the ground ?

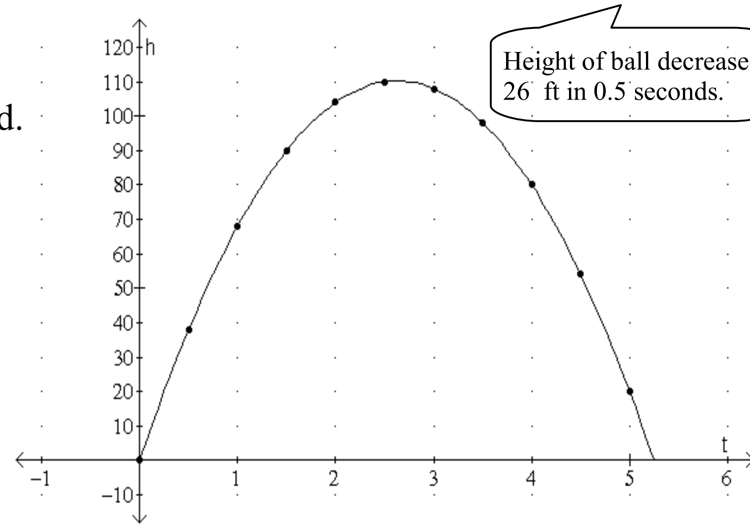
h = 0 feet when the ball hits the ground.

$-16t^2 + 84t = 0$

$-4t(4t - 21) = 0$

$-4t = 0$ or $4t - 21 = 0$

$t = 0$ or $t = 21/4 = 5.25$



The ball hit the ground 5.25 seconds after it was thrown upward from ground level.

h) At what time **t** did the ball reach its maximum height? t_{max} = 2.625 seonds

t_{max} = the average of the two roots
= (0 + 5.25) / 2 = 2.625.

$h_{max} = f(x_{max}) = f(2.625) = 110.25$

i) What was the maximum height of the ball above ground? h_{max} = 110.25 feet

j) At what times **t** was the ball exactly 100 ft above ground? Round to 4 decimal places.

$-16t^2 + 84t = 100$

$4t^2 - 21t = -25$ (Divide both sides by –4. Why?)

$4t^2 - 21t + 25 = 0$

a = 4, b = -21, and c = 25

$$t = \frac{21 \pm \sqrt{441 - 4(100)}}{8} = \frac{21 \pm \sqrt{41}}{8}$$

$t \approx 1.8246$ or $t \approx 3.4254$

When t = 1.8246 seconds or t = 3.4254 seconds, the ball will be 100 feet above ground.

4. From the top of a building, a toy rocket is shot upward with an initial vertical velocity of 132 ft/sec. The formula below gives the height of the rocket above ground in feet at time **t**. Assume that **t** = 0 when the rocket left its launch pad. Round all answers to nearest 0.0001.

132 ft/sec = 90 mph.

h = -16t² + 132t + 60 where **h** = height in feet and **t** = time in seconds.

a) How tall is the building? 60 feet

b) How high above ground was the rocket when **t** = 3 seconds? 312 feet

c) How high above ground was the rocket when **t** = 8 seconds? 92 feet

d) At what time **t** did the rocket hit the ground?

(The negative solution is used to find t_{max}.)

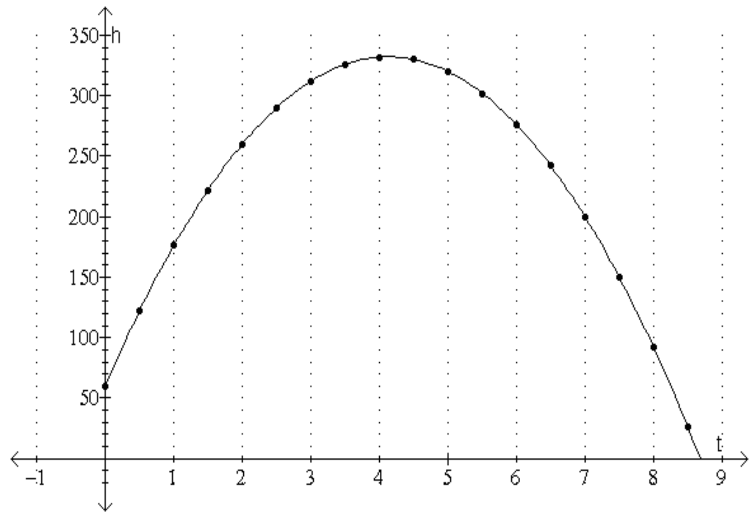
$$-16t^2 + 132t + 60 = 0$$

$$4t^2 - 33t - 15 = 0$$

$$a = 4,\ b = -33, \text{ and } c = -15$$

$$t = \frac{33 \pm \sqrt{1{,}089 - 4(-60)}}{8} = \frac{33 \pm \sqrt{1{,}329}}{8}$$

$$t \approx -0.4319 \text{ or } t \approx 8.6819$$



The rocket hit the ground 8.6819 seconds after launch.

e) At what time **t** did the rocket reach its maximum height **h**?

t_{max} = the average of the two roots
= (-0.4319 + 8.6819) / 2 = 4.125.

t_{max} = 4.125 seonds

f) What was the maximum height reached by the rocket?

$$h_{max} = -16(4.125)^2 + 132(4.125) + 60 = 332.25$$

h_{max} = 332.25 feet

g) At what times **t** was the rocket exactly 300 ft above ground? Round to 4 decimal places.

$$-16t^2 + 132t + 60 = 300$$

$$-16t^2 + 132t - 240 = 0$$

$$4t^2 - 33t + 60 = 0 \quad \text{(Divide both sides by –4. Why?)}$$

$$a = 4,\ b = -33, \text{ and } c = 60$$

$$t = \frac{33 \pm \sqrt{1{,}089 - 4(240)}}{8} = \frac{33 \pm \sqrt{129}}{8}$$

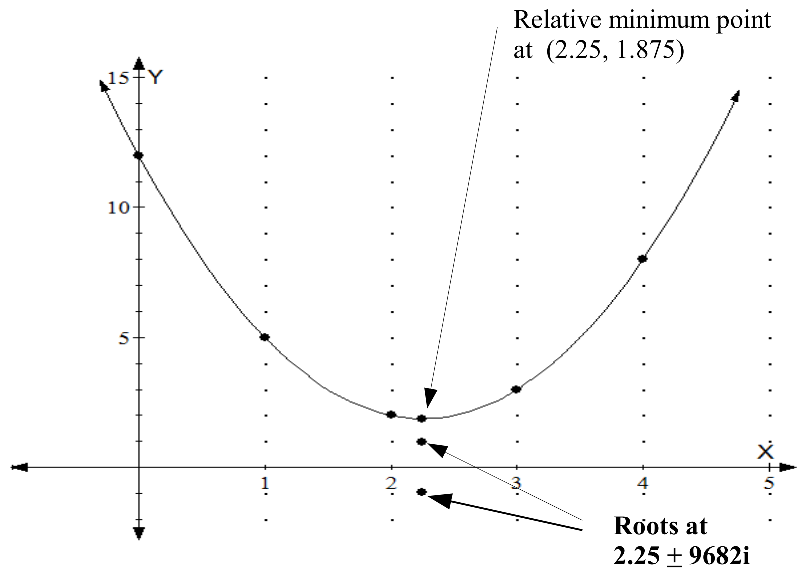
$$t \approx 5.5447 \text{ or } t \approx 2.7053$$

The rocket will be 300 feet above ground when t = 2.7053 or 5.5447 seconds.

5. The graph of $f(x) = y = 2x^2 - 9x + 12$ is shown below.

a) Fill in the table and plot the points on the graph.

x	y
0	12
1	5
2	2
3	3
4	8
5	17



b) The function $y = f(x)$ does not have any real roots. Explain why this is the case.

> The graph of $y = 2x^2 – 9x + 12$ does not intersect the x-axis. Therefore there can be no real values of **x** on the x-axis for which $f(x) = 0$.

c) Use the quadratic formula to find the <u>complex roots</u> of f(x). Write both roots in standard **a + bi** format.

$2x^2 - 9x + 12 = 0$

$a = 2,\ b = -9$, and $c = 12$

> Roots of f(x) :
>
> $x \approx 2.25 + 0.9682i$ or $2.25 – 0.9682i$

$$x = \frac{9 \pm \sqrt{81 - 4(24)}}{4} = \frac{9 \pm \sqrt{-15}}{4} = \frac{9 \pm i\sqrt{15}}{4}$$

$x \approx 2.25 + 0.9682i$ or $x \approx 2.25 - 0.9682i$

d) Find the x-y coordinates of the relative minimum point of f(x) . Round to 4 decimal places.

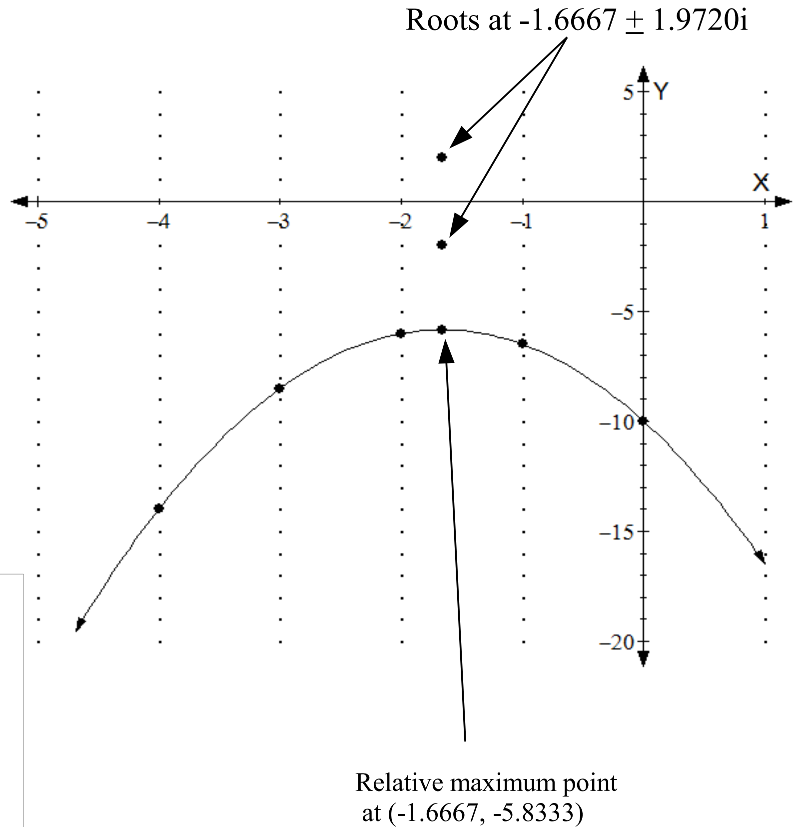
x_{min} = 2.25 and y_{min} = 1.875

e) If the roots of a function are real numbers, we can plot the roots on the x-axis where the graph of f(x) intersects the x-axis. Because the roots of f(x) are complex numbers, we can <u>not</u> plot the roots on the x-axis. If we image the complex number plane layered over the x-y coordinate plane, we can plot the complex roots of f(x). Plot and neatly label the complex roots of f(x) on the x-y coordinate axes above.

6. The graph of $f(x) = y = -1.5x^2 - 5x - 10$ is shown below.

a) Fill in the table and plot the points on the graph.

x	y = f(x)
-5	-22.5
-4	-14
-3	-8.5
-2	-6
-1	-6.5
0	-10



b) The function y = f(x) does not have any real roots. Explain why this is the case.

The graph of $y = -1.5x^2 - 5x - 10$ does not intersect the x-axis. Therefore there can be no real values of **x** on the x-axis for which f(x) = 0.

c) Use the quadratic formula to find the complex roots of f(x). Write both roots in standard **a + bi** format.

$$-1.5x^2 - 5x - 10 = 0$$

$$a = -1.5, \quad b = -5, \text{ and } c = -10$$

$$x = \frac{5 \pm \sqrt{25 - 4(15)}}{-3} = \frac{-5 \pm \sqrt{-35}}{3} = \frac{-5 \pm i\sqrt{35}}{3}$$

$x \approx -1.6667 + 1.9720i$ or $x \approx -1.6667 - 1.9720i$

Roots of f(x):
$x \approx -1.6667 + 1.9720i$ or $-1.6667 - 1.9720i$

d) Find the x-y coordinates of the relative maximum point of f(x) . Round to four decimal places.

x_{max} = -1.6667 and y_{max} = -5.8333

e) If the roots of a function are real numbers, we can plot the roots on the x-axis where the graph of f(x) intersects the x-axis. Because the roots of f(x) are complex numbers, we can not plot the roots on the x-axis. If we image the complex number plane layered over the x-y coordinate plane, we can plot the complex roots of f(x). Plot and neatly label the complex roots of f(x) on the x-y coordinate axes above.