Power of the Quadratic Formula

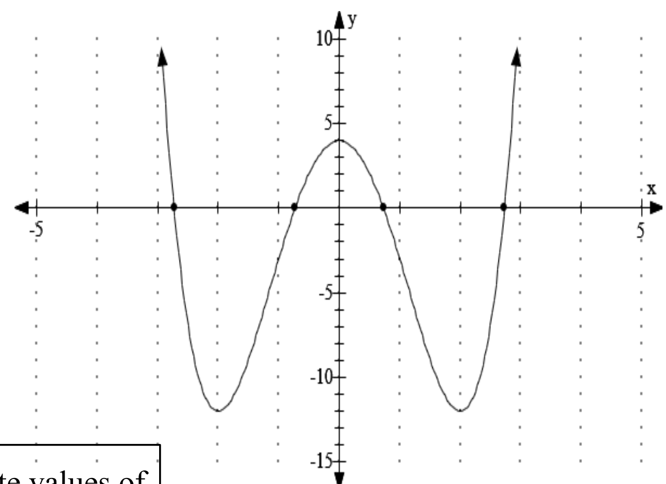
Key

1. Consider the equation $y = x^4 - 8x^2 + 4$. It may be a surprise, but we can use the quadratic formula to find the x-intercepts of graph because the ratio of the two exponents is 2 : 1. Use the quadratic formula to first solve for x^2. Once we know the value of x^2, we can find the value of x itself.

The fundamental theorem of algebra tells us that the polynomial equation $x^4 - 8x^2 + 4 = 0$ has four roots where some of the roots may be multiple roots. Roots of polynomial equations come in two flavors, real or complex. Real roots also come in two flavors, rational and irrational. From Descartes' rule of signs we can deduce that the polynomial has exactly 2 or 0 positive real roots and exactly 2 or 0 negative real roots. We will see that the polynomial $x^4 - 8x^2 + 4$ has exactly 2 positive irrational roots and 2 negative irrational roots.

$y = (x^2)^2 - 8(x^2)^1 + 4$

$a = 1, b = -8, c = 4$



$$x^2 = \frac{8 \pm \sqrt{64 - 4(4)}}{2}$$

$$x^2 = \frac{8 \pm \sqrt{48}}{2} = \frac{8 \pm \sqrt{3 \bullet 16}}{2}$$

$$x^2 = \frac{8 \pm 4\sqrt{3}}{2} = 4 \pm 2\sqrt{3}$$

$x^2 = 7.464101615$ or $x^2 = 0.5358983849$

$x = \pm 2.732050808$ or $x = \pm 0.7320508076$

> There are four irrational roots with approximate values of $x = \pm 2.732050808$ or $x = \pm 0.7320508076$ ✓

Tip to check answers for TI graphing calculator users:

The <ANS> key is a wonderful convenience. Follow the steps below to plug 2.732050808 into the equation $x^4 - 8x^2 + 4$.

1. Enter √(4 + 2√(3)). Make sure to press the <ENTER> key.

2. Enter the expression ANS^4 - 8ANS^2 + 4

3. The output from step (2) above should equal $2 * 10^{-12} = 0.000\,000\,000\,002 \approx 0$.

Note:

If 2.732050808^4 – 8*2.732050808^2 + 4 is entered, the output = $1.632 * 10^{-8} \approx 0$.

Keep in mind that many numerical results have a rounding error! Only humans know the exact answer.

2. Consider the equation $y = x^4 - 8x^2 - 4$. We can use the quadratic formula to find the x-intercepts of graph because the ratio of the two exponents is 2 : 1. Use the quadratic formula to first solve for x^2. Once we know the value of x^2, we can find the value of x itself.

The fundamental theorem of algebra tells us that the polynomial equation $x^4 - 8x^2 - 4 = 0$ has four roots where some of the roots may be multiple roots. Roots of polynomial equations come in two flavors, real or complex. Real roots also come in two flavors, rational and irrational. From Descartes' rule of signs we can deduce that the polynomial has exactly 1 positive real root and exactly 1 negative real root. Therefore the other two roots of $x^4 - 8x^2 - 4$ must be complex roots. How can one tell by inspection of the equation $y = x^4 - 8x^2 - 4$ that the function $y = f(x)$ is an even function or the graph has symmetry with respect to the y-axis?

$y = (x^2)^2 - 8(x^2)^1 - 4$

$a = 1, b = -8, c = -4$

$$x^2 = \frac{8 \pm \sqrt{64 - 4(-4)}}{2}$$

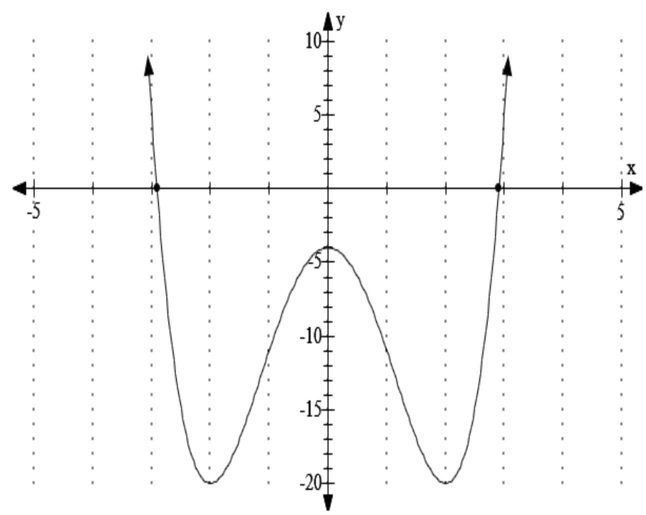
$$x^2 = \frac{8 \pm \sqrt{80}}{2} = \frac{8 \pm \sqrt{5 \bullet 16}}{2}$$

$$x^2 = \frac{8 \pm 4\sqrt{5}}{2} = 4 \pm 2\sqrt{5}$$

$x^2 = 8.472135955$ or $x^2 = -0.472135955$

$x = \pm\sqrt{4 + 2\sqrt{5}}$ or $x = \pm i\sqrt{|4 - 2\sqrt{5}|}$

$x = \pm\ 2.91069338$ or $x = \pm\ 0.6871214994i$



There are two irrational and two complex roots with approximate values of $x = \pm\ 2.91069338$ or $x = \pm\ 0.687121499i$ ✓

Follow the steps below to plug in 0.687121499i into the equation $x^4 - 8x^2 - 4$.

1. Press the <MODE> button and set the number mode to <a + bi>.

2 Enter 0.687121499i . (**i** is the button above the decimal point key) Make sure to press the <ENTER> key.

3. Enter the expression ANS^4 - 8ANS^2 - 4

4. The output from step (3) above should equal $-5.47 * 10^{-9} = -0.000\ 000\ 005\ 47 \approx 0$.

Keep in mind that many numerical results will have a rounding error! Only humans know the exact answer.

3. Consider the equation $y = x^{2/3} - 2x^{1/3} - 3$. It may be a surprise, but we can use the quadratic formula to find the x-intercepts of graph because the ratio of the two exponents is 2 : 1. Use the quadratic formula to first solve for $x^{1/3}$. Once we know the value of $x^{1/3}$, we can find the value of x itself. (Descartes' rule of signs only applies to polynomials with whole number exponents.)

$$x^{2/3} - 2x^{1/3} - 3 = 0$$

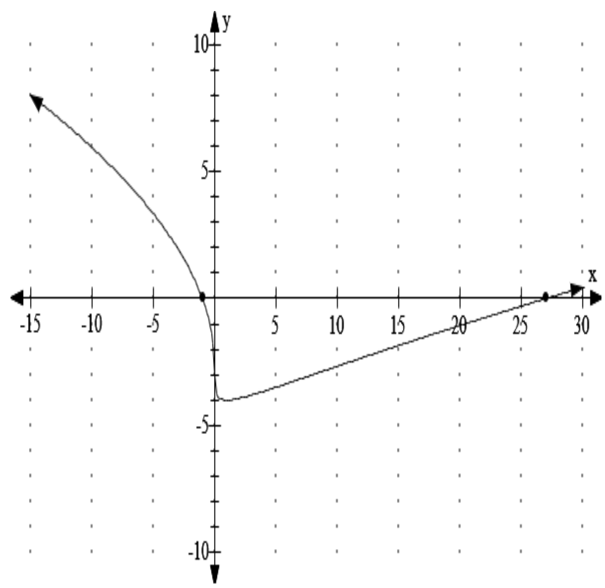
$$\left(x^{1/3}\right)^2 - 2\left(x^{1/3}\right)^1 - 3 = 0$$

$a = 1$, $b = -2$, $c = -3$

$$x^{1/3} = \frac{2 \pm \sqrt{4-4(-3)}}{2} = \frac{2 \pm \sqrt{16}}{2}$$

$$x^{1/3} = 3 \text{ or } x^{1/3} = -1$$

$$x = 3^3 = 27 \text{ or } x = (-1)^3 = -1$$



There are two rational roots with values of $x = -1$ or $x = 27$.

4. Consider the equation $y = x^{3/2} - 2x^{3/4} - 3$. We can use the quadratic formula to find the x-intercepts of graph because the ratio of the two exponents is 2 : 1. Use the quadratic formula to first solve for $x^{3/4.}$ Once we know the value of $x^{4/3}$, we can find the value of x itself.

$$x^{3/2} - 2x^{3/4} - 9 = 0$$

$$\left(x^{3/4}\right)^2 - 2\left(x^{3/4}\right)^1 - 9 = 0$$

$a = 1$, $b = -2$, and $c = -9$

$$x^{3/4} = \frac{2 \pm \sqrt{4-4(-9)}}{2} = \frac{2 \pm \sqrt{40}}{2}$$

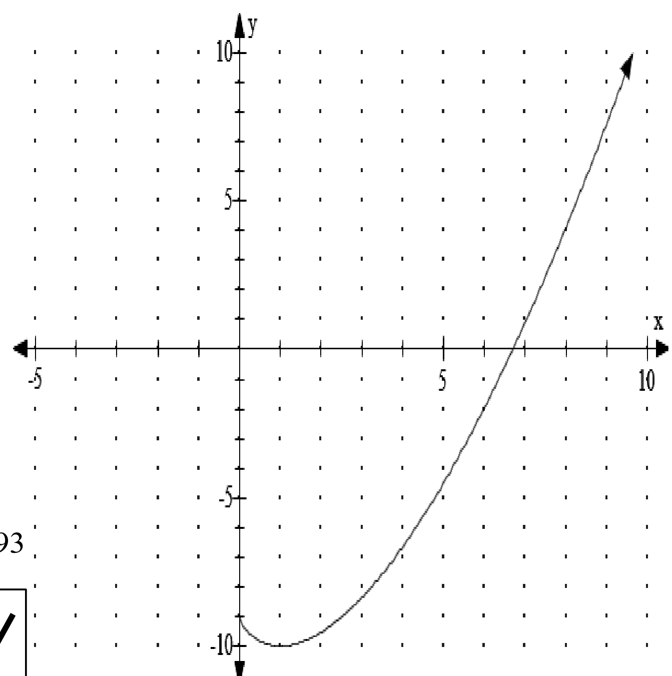
$$x^{3/4} = \frac{2 \pm \sqrt{4 \bullet 10}}{2} = \frac{2 \pm 2\sqrt{10}}{2}$$

$$x^{3/4} = 1 + \sqrt{10} \text{ or } x = 1 - \sqrt{10}$$

$$x^{3/4} = 4.16227766 \text{ or } -2.16227766$$

$x^{3/4} = -2.16227766$ is not possible. Why?

$$x = \left(x^{3/4}\right)^{4/3} = 4.16227766^{4/3} = 6.695372293$$



There is one irrational root with an approximate value of $x = 6.695372293$.

5. Consider the equation y = 2x – 9√(x) + 4. It may be a surprise, but we can use the quadratic formula to find the x-intercepts of graph because the ratio of the two exponents is 2 : 1. Use the quadratic formula to first solve for $x^{1/2}$. Once we know the value of $x^{1/2}$, we can find the value of x itself.

It would be faster to solve the equation by factoring since 2x – 9√(x) + 4 = (2√(x) - 1)(√(x) – 4). Students should always first try the factoring method to solve the equation. If you don't see how to factor the expression after a brief amount of time, use the quadratic formula. It always works.

$$y = 2x - 9\sqrt{x} + 4$$

$$y = 2\left(x^{1/2}\right)^2 - 9\left(x^{1/2}\right)^1 + 4$$

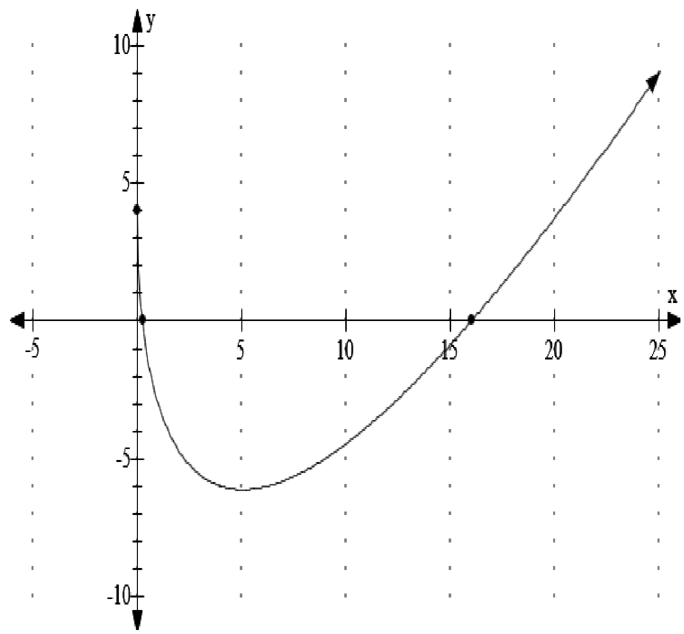
$$a = 2,\ b = -9,\ c = 4$$

$$x^{1/2} = \frac{9 \pm \sqrt{49}}{4} = \frac{9 \pm 7}{4}$$

$$x^{1/2} = 4 \quad \text{or} \quad x^{1/2} = \frac{1}{2}$$

$$\left(x^{1/2}\right)^2 = 4^2 \quad \text{or} \quad \left(x^{1/2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$x = 16 \quad \text{or} \quad x = \frac{1}{4}$$



There are two rational roots with values of x = 16 or x = 1/4. ✓

Some things to think about.

What is it about the equation y = 2x – 9√(x) + 4 that causes the graph to have no points to the left of the y-axis?

Consider the equation y = 2|x| – 9√(| x |) + 4 .
Can you predict what the graph of this equation looks like?
Can you find all roots of the equation without any additional work?

6. Consider the implicitly defined relation $2x^2 - 3xy = 4y - 2$. Even though b and c are y-variable expressions, we can use the quadratic formula to solve the equation for x. Of course, we can also solve the equation for y. All three equations express the relationship between the x-coordinate and y-coordinate for every point (x,y) on the graph, but in a different way. It is truly amazing to see that the output graphs of all three equations are identical.

Solve $2x^2 - 3xy = 4y - 2$ for y.

$$2x^2 + 2 = 3xy + 4y$$
$$2x^2 + 2 = y(3x + 4)$$
$$y = \frac{2x^2 + 2}{3x + 4}$$

Observations:

Since the numerator can never equal zero, $y = f(x)$ has no real roots and the graph can not cross the x-axis.

When $x = -4/3$, the denominator equals zero. Therefore the graph of $y = f(x)$ must have a vertical asymptote at $x = -4/3$.

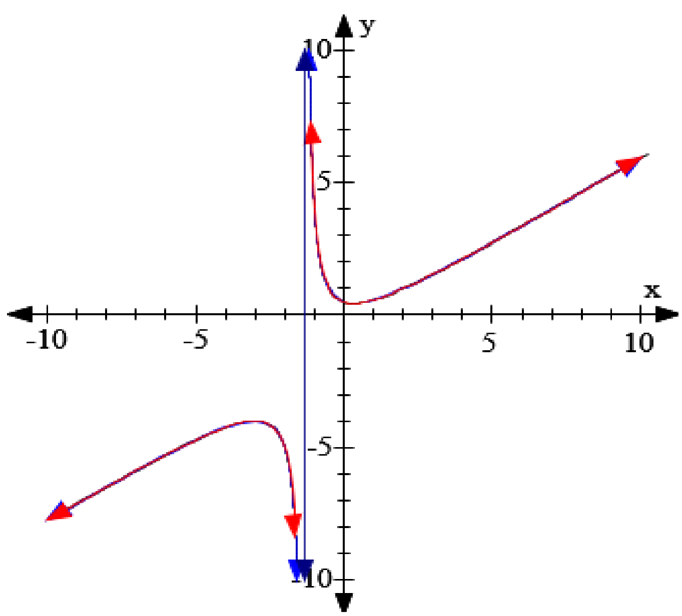
Use quadratic formula to solve $2x^2 - 3xy = 4y - 2$ for x.

$$2x^2 - 3xy - 4y + 2 = 0$$
$$a = 2,\ b = -3y,\ c = -4y + 2$$

$$x = \frac{3y \pm \sqrt{9y^2 - 4(-8y + 4)}}{4}$$
$$x = \frac{3y \pm \sqrt{9y^2 + 32y - 16}}{4}$$

Observation:

The $\pm$ operator expresses x as two explicit functions of y. Therefore for every valid input value of y, there are two output values for x. As y changes from y_{min} to y_{max}, we can see two horizontal output values x for every valid element in the domain of y.



7. Consider the implicitly defined relation $0.5y^2 – (4x – 5)y + x – 12 = 0$. Even though b and c are x-variable expressions, we can use the quadratic formula to solve the equation for y. Of course, we can also solve the equation for x. All three equations express the relationship between the x-coordinate and y-coordinate for every point (x,y) on the graph, but in a different way. It is truly amazing to see that the output graphs of all three equations are identical.

Use the quadratic formula to solve
$0.5y^2 – (4x – 5)y + x – 12 = 0$
for y.

$0.5y^2 - (4x - 5)y + x - 12 = 0$

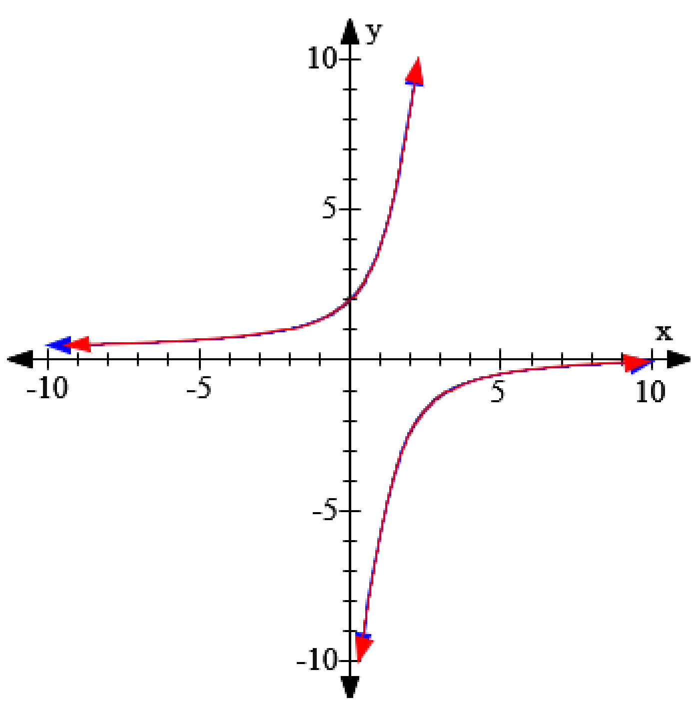
$a = 0.5,\ b = -(4x - 5),\ c = x - 12$

$y = 4x - 5 \pm\sqrt{(4x-5)^2 - 4(0.5(x-12))}$

$y = 4x - 5 \pm\sqrt{16x^2 - 40x + 25 - 2(x-12)}$

$y = 4x - 5 \pm\sqrt{16x^2 - 40x + 25 - 2x + 24}$

$y = 4x - 5 \pm\sqrt{16x^2 - 42x + 49}$



Solve $0.5y^2 – (4x – 5)y + x – 12 = 0$
for x.

$0.5y^2 - (4x - 5)y + x - 12 = 0$

$0.5y^2 - 4xy + 5y + x - 12 = 0$

$x - 4xy = -(0.5y^2 + 5y - 12)$

$x(1 - 4y) = -(0.5y^2 + 5y - 12)$

$x = \dfrac{0.5y^2 + 5y - 12}{4y - 1}$

Observations:

The equation $0.5y^2 + 5y – 12 = 0$ has roots $y = 2$ and $y = -12$. Therefore the graph of $x = f(y)$ has y-intercepts at (0, 2) and (0, -12).

When $4y – 1 = 0$, $y = ¼$. Therefore the graph of $x = f(y)$ must have a horizontal asymptote at $y = ¼$.