Observations About the Roots of a Polynomial

Background: In many cases, the problem of solving an equation can be reduced to finding the roots of a polynomial. An example of this technique is shown below.

Solve: $\frac{2x}{x+1} - \frac{x}{4} = x^3$ ====> $4(x+1)\left(\frac{2x}{x+1} - \frac{x}{4}\right) = 4(x+1)x^3$

Warning! Multiplying both sides of an an equation by an expression that contains a variable can **create** solutions that are **not** solutions of the original equation. Therefore you need to check all solutions with the original equation.

$$8x - x(x+1) = x^3(4x+4)$$
$$8x - x^2 - x = 4x^4 + 4x^3$$
$$4x^4 + 4x^3 + x^2 - 7x = 0$$
$$x(4x^3 + 4x^2 + x - 7) = 0$$

Warning! Dividing both sides of an equation by an expression that contains a variable can **destroy** solutions of the original equation. **Never divide** both sides of an equation by an expression that contains a variable.

The last equation tells us that $x = 0$ is a root. Our problem reduces to finding the roots of $4x^3 + 4x^2 + x - 7$.

Fundamental Theorem of Algebra: Every polynomial of degree **n** ($n > 0$) with real or complex coefficients has exactly **n** roots. Multiple roots are counted individually. (Gauss 1777 – 1855)

Example: By the fundamental theorem of algebra, we know that $2x^3 + 7$ has three roots and $13x^{1,927} + 22x^5 - 6ix^2 + 43$ has 1,927 roots! Finding these roots would be a daunting task indeed!

Nature of the Roots: The roots of a polynomial are rational numbers, irrational numbers, or complex numbers.

Conjugate Pairs Theorem: If all of the coefficients of a polynomial P(x) are real numbers, then all complex roots of P(x) appear in conjugate pairs. If $P(a + bi) = 0$, then $P(a - bi) = 0$.

Roots of P(x) and P(-x): The graphs of P(x) and P(-x) are reflection images of each other over the y-axis. Therefore the roots of P(x) and the roots of P(-x) have opposite signs. In order to use Descartes' rule of signs to determine the number of negative roots of P(x), we apply Descartes' rule of signs to P(-x). See below.

Descartes' Rule of Signs: If P(x) is a polynomial with all real coefficients, the number of positive real roots of P(x) equals the number of sign changes in the coefficients of P(x) or less than that by an even number. The number of negative real roots of P(x) equals the number of positive roots of P(-x). (Rene' Descartes 1596-1650)

Example 1: Consider $P(x) = 3x^4 - 10x^3 - 12x^2 + 6x - 4$ and $P(-x) = 3x^4 + 10x^3 - 12x^2 - 6x - 4$

P(x) has 3 sign changes and therefore P(x) has exactly 3 or 1 positive real roots. P(-x) has one sign change and therefore P(-x) has exactly one positive real root or P(x) has exactly one negative real root.

Example 2: Consider $P(x) = x^{10} + 1024$ and $P(-x) = x^{10} + 1024$. (If $x^{10} + 1024 = 0$, x is a tenth root of -1024.)

Both P(x) and P(-x) have zero sign changes and therefore P(x) has no positive or negative real roots. Therefore the ten tenth roots of -1,024 are complex numbers which appear in conjugate pairs. From De Moivre's theorem, it follows that all roots lie on a circle centered at (0,0) and radius of 2. The angular spacing between the ten tenth roots equals 36^0 and the first tenth root of $-1,024 = 2\text{Cos}(18^0) + 2\text{Sin}(18^0)i$.

Example 3: Consider $P(x) = 6x^{12} + 5x^9 - x^6 - x^5 + x^2 - x + 40$ and $P(-x) = 6x^{12} - 5x^9 - x^6 + x^5 + x^2 + x + 40$. P(x) has four sign changes and therefore P(x) has 4, 2 or 0 real roots. P(-x) has 2 sign changes and therefore P(x) has 2 or 0 negative real roots.

Rational Root Theorem: Let P(x) be a polynomial with <u>all integer coefficients</u> and the constant term does **not** equal zero. If p/q is a rational root of P(x), then q is a divisor of the leading coefficient and p is a divisor of the constant term of P(x).

Example 1: Consider $P(x) = x^{67} - 219x^{50} + 10x^{21} + 12x - 60$.

If $\frac{p}{q}$ is a root of P(x) then p is a factor of 60 and q is factor of 1.

Therefore the only possible rational roots of P(x) are $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm10. \pm12, \pm15, \pm20, \pm30,$ and ±60.

> Keep in mind that there is no guarantee that at least one of the possible rational roots is a root. In many cases, none of the possible rational roots is a root.

Example 2: Consider $P(x) = 6x^7 + 19x^5 - 10x^4 - 14$.

If $\frac{p}{q}$ is a root of P(x) then p is a factor of 14 and q is factor of 6.

Therefore the only possible rational roots of P(x) are of the form $\pm\frac{1, 2, 7, 14}{1, 2, 3, 6}$.

Example 3: Consider $P(x) = 24x^4 + 2x^3 - 242x^2 + 141x + 270$.

Factored: $P(x) = (3x + 10)(4x - 9)(2x^2 - 2x - 3)$

Roots of P(x): $-\frac{10}{3}, \frac{9}{4}, \frac{1 \pm \sqrt{7}}{2}$

> Use the quadratic formula to find the two irrational roots.

If $\frac{p}{q}$ is a root of P(x) then p is a factor of 270 and q is factor of 24.

Therefore the only possible rational roots of P(x) are of the form

$$\pm\frac{1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 90, 135, 270}{1, 2, 3, 4, 6, 8,12, 24}$$

> Notice that when multiplying the factors of the polynomial, the denominators of the rational roots contribute to the the leading coefficient of P(x) and the numerators of the rational roots contribute to the constant term of P(x).