

## 2D Geometric Transformations and Matrices

**Background:** Objects are drawn and moved in 2D space and 3D space on a computer screen by multiplying matrices. Generally speaking, computer animation is achieved as follows by repeating steps 1, 2, and 3 below.

- 1) Use the coordinates of the current geometric object or shape to calculate the coordinates of a new geometric object. The coordinates of a new object are calculated by multiplying the coordinates of the current object by a matrix so that the product will give us the coordinates of an object to that has been translated, rotated about a point, reflected over a line or plane, horizontally stretched/shrunk, or vertically stretched/shrunk.
- 2) Erase the object from the screen.
- 3) Use the coordinates of the new object to draw the object on the screen and then make the coordinates of the current object equal to the coordinates of the new object.

**Basic 2D Transformation Matrices:** The basic matrices listed below are the matrices that computer programmers would use to construct a geometric transformation matrix which would cause a figure to be translated in a specific direction, rotated about a point, reflected over a line, stretched about a point, shrunk about a point or sheared. All 2D affine geometric transformations can be described as a product of two or more of the basic 3 x 3 transformation matrices. Each of these matrices has an inverse matrix which makes it easy for a computer programmer to reverse or undo a transformation of a figure.

**Mapping of Preimage Points to Image Points:** The mapping of a preimage point in the x-y coordinate plane to an image point in the x-y coordinate plane is accomplished by multiplying the preimage point by a geometric transformation matrix. In the listing below, preimage points (x, y) are represented by a 3 x 1 matrix to the right of the transformation matrix. The resulting image point is represented by the 3 x 1 matrix to the left of the equal sign.

### The Basic Transformation Matrices

#### Translation or Slide

$$\begin{bmatrix} x+j \\ y+k \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(x+j, y+k) \leftarrow (x, y)$$

#### Reflection over x-axis

$$\begin{bmatrix} x \\ -y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(x, -y) \leftarrow (x, y)$$

#### Reflection over y-axis

$$\begin{bmatrix} -x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(-x, y) \leftarrow (x, y)$$

#### Shear

$$\begin{bmatrix} x+jy \\ y+kx \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & j & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(x+jy, y+kx) \leftarrow (x, y)$$

#### Size Transformation from (0, 0)

$$\begin{bmatrix} jx \\ ky \\ 1 \end{bmatrix} = \begin{bmatrix} j & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(jx, ky) \leftarrow (x, y)$$

#### Rotation about (0,0)

$$\begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta)) \leftarrow (x, y)$$

If  $|j| > 1$ , figure is expanded horizontally by a factor of  $j$ .  
 If  $|j| < 1$ , figure is shrunk horizontally by a factor of  $j$ .  
 If  $|k| > 1$ , figure is expanded vertically by a factor of  $k$ .  
 If  $|k| < 1$ , figure is shrunk vertically by a factor of  $k$ .

If  $\theta > 0^\circ$ , the figure is rotated CCW  $\theta$  degrees about (0,0).  
 If  $\theta < 0^\circ$ , the figure is rotated CW  $\theta$  degrees about (0,0).

# Matrix Multiplication and Geometric Linear Transformations

A linear geometric transformation of a figure is accomplished by performing a series of basic transformations to the figure in some specific order. The order in which two transformations are performed is important because geometric transformation operations and matrix multiplication are generally non commutative. In order to achieve desired results, one needs to pay attention to the order in which the operations are performed.

A linear transformation matrix is constructed by calculating the product of basic geometric transformation matrices. Each matrix in the matrix product is a basic geometric transformation matrix which corresponds to a basic geometric transformation. Due to how matrix multiplication works and the fact that the geometric transformation matrix will be multiplied by a 3-row by n-column preimage matrix of points, the matrices must be listed in right-to-left order in the product of transformation matrices. Because matrix multiplication is associative, it is not necessary to include any grouping symbols in the matrix product expression. The following examples illustrate how to use matrix based transformations to do a geometric transformation on a preimage figure.

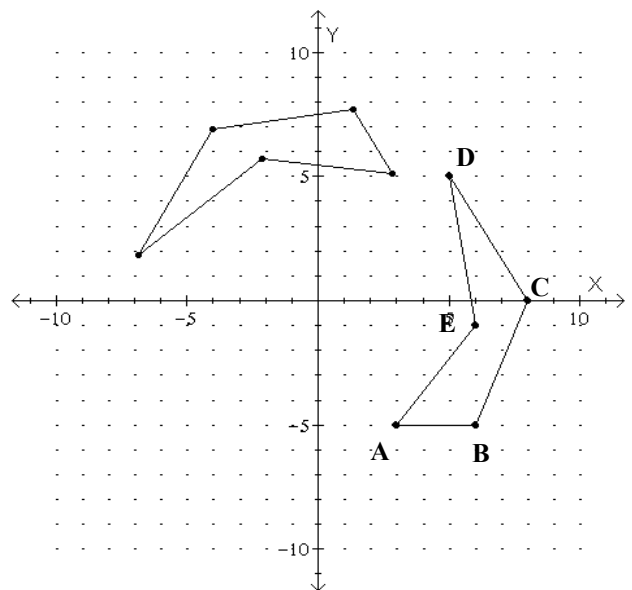
**Example 1:** Rotate polygon ABCDE 120° counterclockwise about the origin.

$$[P] = \begin{bmatrix} 3 & 6 & 8 & 5 & 6 & 3 \\ -5 & -5 & 0 & 5 & -1 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Preimage matrix of nonconvex} \\ \text{polygon to the right.} \end{array}$$

$$[T] = \begin{bmatrix} -.500 & -.866 & 0 \\ .866 & -.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Rotation matrix (120° CCW} \\ \text{rotation about (0,0).} \\ \text{Sin(120°) = .866, Cos(120°) = 0.5} \end{array}$$

$$[P'] = \begin{bmatrix} 2.8 & 1.3 & -4.0 & -6.8 & 2.8 \\ 5.1 & 7.7 & 6.9 & 1.8 & 5.1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Image matrix of non-} \\ \text{convex polygon to} \\ \text{the left.} \end{array}$$

$$[P'] = [T][P]$$



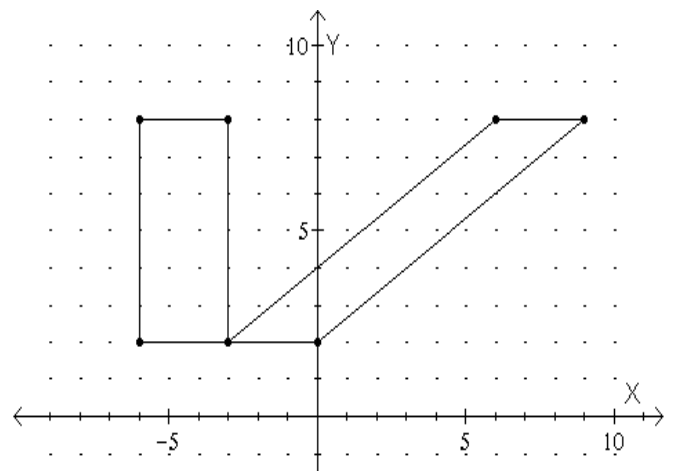
**Example 2:** Horizontal shear transformation with horizontal shear factor  $j = 1.5$

$$[P] = \begin{bmatrix} -6 & -3 & -3 & -6 & -6 \\ 2 & 2 & 8 & 8 & 2 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Preimage matrix of rectangle.} \end{array}$$

$$[T] = \begin{bmatrix} 1 & 1.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Shear transformation matrix} \end{array}$$

$$[P'] = \begin{bmatrix} -3.0 & 0.0 & 9.0 & 6.0 & -3.0 \\ 2.0 & 2.0 & 8.0 & 8.0 & 2.0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Image matrix of} \\ \text{parallelogram.} \end{array}$$

$$[P'] = [T][P]$$



**Example 3:** Expand by a factor of 1.5 and rotate 90° clockwise about (-4, -6)

Preimage matrix of smaller flag to the left.

Rotation matrix: (90° CW rotation About (0,0). Notice that  $\text{Sin}(-90^\circ) = -1$  and  $\text{Cos}(-90^\circ) = 0$

$$[P] = \begin{bmatrix} -4.0 & -6.0 & -2.0 & -5.5 \\ -6.0 & 6.0 & 6.0 & 3.0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[T0] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation matrix

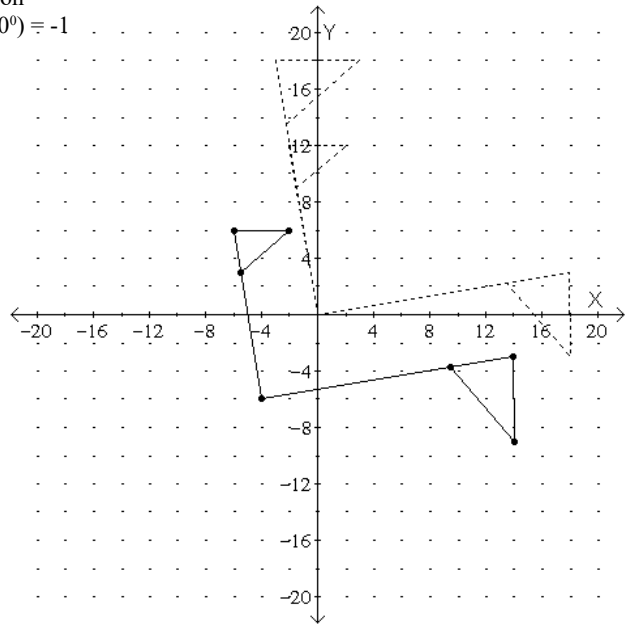
Translation matrix

$$[T1] = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T2] = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T3] = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Size transformation matrix with scale factor of 1.5 and expanded about (0, 0).



$$[T] = [T2][T0][T3][T1] = \begin{bmatrix} 0 & 1.5 & 5 \\ -1.5 & 0 & -12 \\ 0 & 0 & 1 \end{bmatrix} = \text{Geometric transformation matrix that causes the figure to be enlarged by a factor of 1.5 and rotated } 90^\circ \text{ CW about } (-4, -6).$$

$$[P'] = [T2][T0][T3][T1][P] = [T][P] \quad \begin{bmatrix} -4.0 & 14.0 & 14.0 & 9.5 \\ -6.0 & -3.0 & -9.0 & -3.7 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1.5 & 5 \\ -1.5 & 0 & -12 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4.0 & -6.0 & -2.0 & -5.5 \\ -6.0 & 6.0 & 6.0 & 3.0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$[P'] \qquad \qquad [T] \qquad \qquad [P]$

**Comments:**

- [T1] and [T2] are inverse matrices. You should verify that  $[T1][T2] = [T2][T1]$  equals the identity matrix.
- Let [I0] equal the inverse matrix of [T0]. [I0] is a 90° counterclockwise rotation about (0, 0). Use the facts that  $\text{Sin}(90^\circ) = 1$  and  $\text{Cos}(90^\circ) = 0$  to find [I0]. Then verify that  $[T0][I0] = [I0][T0] =$  the identity matrix.
- Let [I3] equal the inverse of the size transformation matrix T3. [I3] is a size transformation matrix with scale factors j and k both set to 2/3. Find [I3] and verify that  $[T3][I3] = [I3][T3] =$  the identity matrix.
- The inverse of [T] is the product of inverse matrices, but multiplied in reverse order. The inverse of [T] equals  $[I1][I3][I0][I2]$ . This can be easily verified by observing the following:

$$[I1][I3][I0][I2] [T2][T0][T3][T1] = [T2][T0][T3][T1] [I1][I3][I0][I2] = \text{the identity matrix.}$$

Like multiplication of real numbers, matrix multiplication is associative. When we multiply middle pairs of matrices, the product collapses to the identity matrix. Consider the following product of real numbers:  
 $5/4(3/2(5*1/5)2/3)4/5 = 5/4(3/2 (1)2/3)4/5 = 5/4(1)4/5 = 1$

**Example 4:** Reflect the upper nonconvex polygon over the line  $y = x + 4$ .

Preimage matrix of upper nonconvex polygon .

$$[P] = \begin{bmatrix} 3 & -4 & -7 & -3 & -3 & 3 \\ 8 & 9 & 4 & -3 & 6 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Translation matrix

$$[T0] = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation matrix

$$[T1] = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection over x-axis

$$[T4] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation matrix (45° CW rotation about (0,0) .

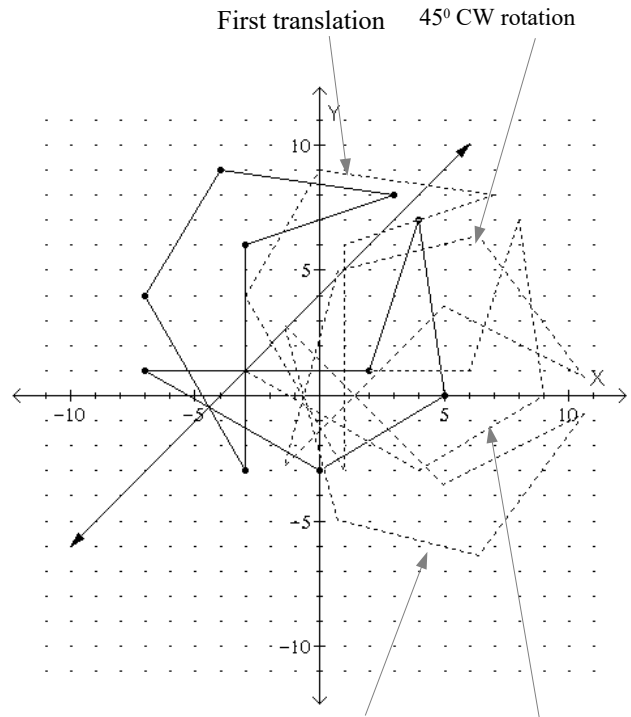
$$\sin(-45^\circ) = -.707, \cos(-45^\circ) = .707$$

Rotation matrix (45° CCW rotation about (0,0) .

$$\sin(45^\circ) = .707, \cos(45^\circ) = .707$$

$$[T2] = \begin{bmatrix} .707 & .707 & 0 \\ -.707 & .707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T3] = \begin{bmatrix} .707 & -.707 & 0 \\ .707 & .707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection over x-axis after 45° CW rotation

$$[T] = [T1][T3][T4][T2][T0] = \begin{bmatrix} 0.0 & 1.0 & -4.0 \\ 1.0 & 0 & 4.0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Geometric transformation matrix that causes the polygon to be reflected over the line } y = x + 4.$$

$$[P'] = [T1][T3][T4][T2][T0][P] = [T][P] \quad \begin{bmatrix} 0.0 & 1.0 & -4.0 \\ 1.0 & 0 & 4.0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 0 & -7 & 2 & 4 \\ 7 & 0 & -3 & 1 & 1 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & -7 & -3 & -3 & 3 \\ 8 & 9 & 4 & -3 & 6 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$[P']$ 
 $[T]$ 
 $[P]$

**Comments:**

1.  $[T0]$  and  $[T1]$  are inverse matrices. You should verify that  $[T0][T1] = [T1][T0]$  equals the identity matrix.
2. Show why matrices  $[T2]$  and  $[T3]$  are inverse matrices.
3. What is the inverse matrix of matrix  $[T4]$  ?
4. The inverse of  $[T]$  is the product of inverse matrices, but multiplied in reverse order. The inverse of  $[T] = [I0][I2][I4][I3][T1]$ . This can be easily verified by observing the following:

$$[I0][I2][I4][I3][I1] [T1][T3][T4][T2][T0] = [T1][T3][T4][T2][T0] [I0][I2][I4][I3][I1] = \text{the identity matrix.}$$

(Refer to the comments on previous page.)

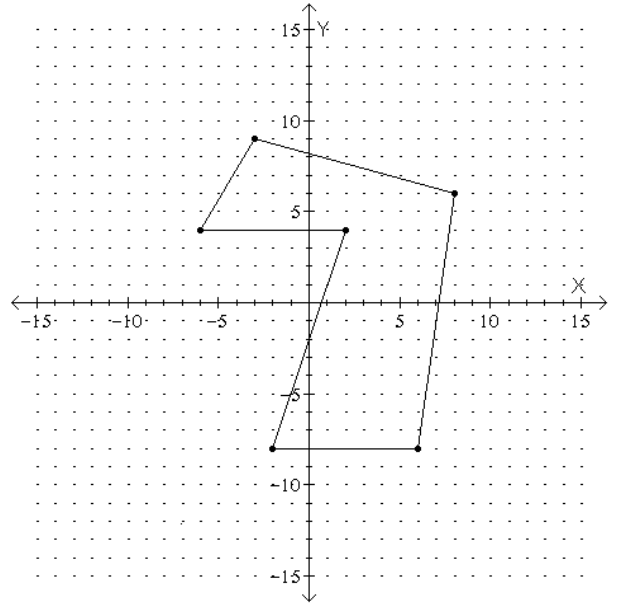
# Geometric Transformation Exercises

Name \_\_\_\_\_

**Exercise 1:** Multiply the transformation matrix  $[T]$  by the preimage matrix  $[P]$ . Use  $[P']$  to draw the transformation image.

$$\begin{array}{c} \text{Image matrix} \\ [P'] \end{array} = \begin{array}{c} \text{Translate} \\ [T] \end{array} \begin{array}{c} \text{Preimage matrix} \\ [P] \end{array}$$

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & -3 & 8 & 6 & -2 & 2 & -6 \\ 4 & 9 & 6 & -8 & -8 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



**Exercise 2 :**

$$\begin{array}{c} \text{Transformation} \\ \text{matrix} \\ [T] \end{array} = \begin{array}{c} \text{Reflect over} \\ \text{x-axis} \end{array} \begin{array}{c} \text{Translate} \end{array}$$

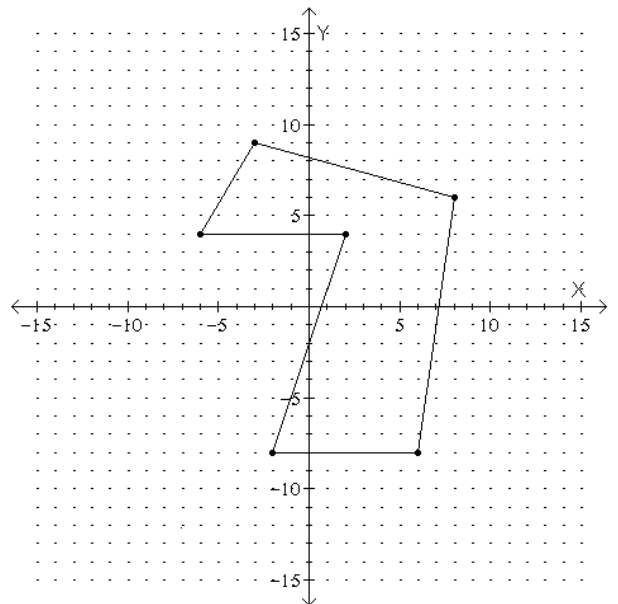
$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & -1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

The figure will be first translated and then reflected over the x-axis. Transformation matrices must be listed in right to left order.

Multiply the transformation matrix  $[T]$  by the preimage matrix  $[P]$ . Use  $[P']$  to draw the transformation image.

$$\begin{array}{c} \text{Image matrix} \\ [P'] \end{array} = \begin{array}{c} \text{Transformation} \\ \text{matrix} \\ [T] \end{array} \begin{array}{c} \text{Preimage matrix} \\ [P] \end{array}$$

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & -1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & -3 & 8 & 6 & -2 & 2 & -6 \\ 4 & 9 & 6 & -8 & -8 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



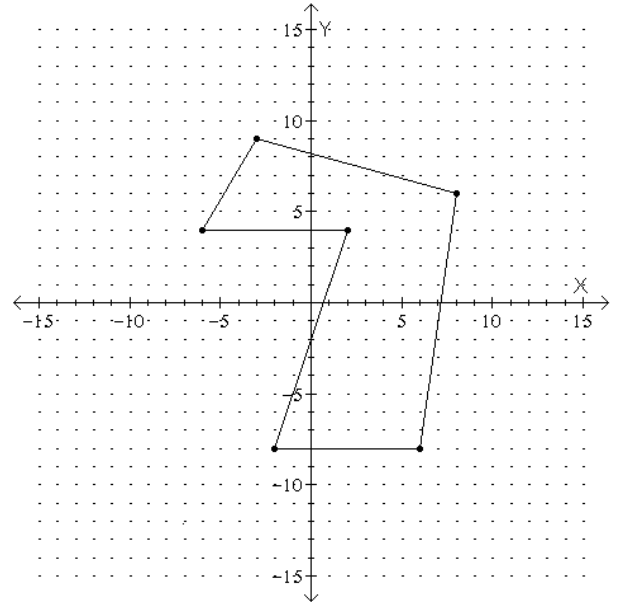
**Exercise 3:**

$$[T] = \begin{matrix} \text{Transformation} \\ \text{matrix} \end{matrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{matrix} \text{Translate} \end{matrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{Reflect over} \\ \text{x-axis} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The figure will be first reflected over the x-axis and then translated. Matrices must be listed in right to left order.)

Multiply the transformation matrix [T] by the preimage matrix [P]. Use [P'] to draw the transformation image.

$$\begin{matrix} \text{Image matrix} \\ [P'] \end{matrix} \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{matrix} \text{Transformation} \\ \text{matrix} \\ [T] \end{matrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{Preimage matrix} \\ [P] \end{matrix} \begin{bmatrix} -6 & -3 & 8 & 6 & -2 & 2 & -6 \\ 4 & 9 & 6 & -8 & -8 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



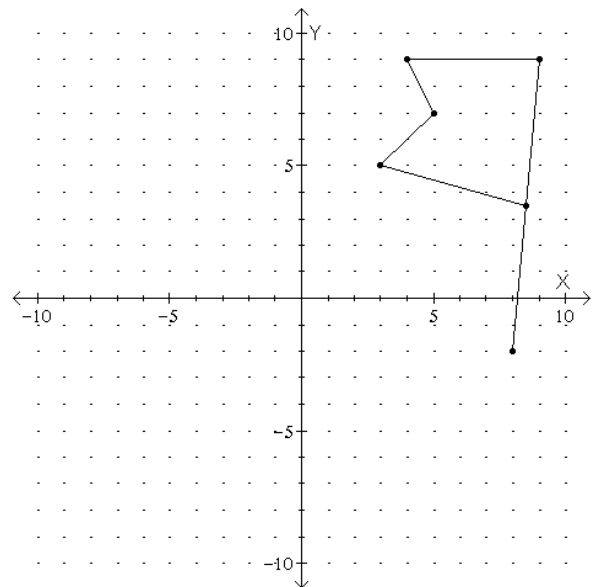
**Exercise 4:**  $[T] = \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & -10 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Translate back to center of rotation
Rotate 90° CCW about (0,0)
Translate to center of rotation to (0, 0)

Transformation matrices must be listed in right to left order.

Multiply the transformation matrix [T] by the preimage matrix [P]. Use [P'] to draw the image.

$$\begin{matrix} \text{Image matrix} \\ [P'] \end{matrix} \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{matrix} \text{Transformation} \\ \text{matrix} \\ [T] \end{matrix} \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & -10 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{Preimage matrix} \\ [P] \end{matrix} \begin{bmatrix} 8 & 9 & 4 & 5 & 3 & 8.5 \\ -2 & 9 & 9 & 7 & 5 & 3.5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



5. Write a translation matrix that would cause a figure to be slid 10 units to the left and 5 units up.

6. Write the inverse of the matrix from exercise (5) above. Verify that it is the inverse matrix.

7. The matrix to the right could be used to rotate a figure  $180^\circ$  about  $(0,0)$ .  
Write the inverse of this matrix. Verify that it is the inverse matrix.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Write a transformation matrix that would cause a figure to be reflected over the line  $y = x$ .  
Hint: Recall that reflecting the point  $(x,y)$  over the line  $y = x$  is equivalent to mapping  $(x,y)$  to  $(y,x)$ .

9. Write the inverse of the matrix from exercise (8) above and verify that it is the inverse matrix.

10. Write a transformation matrix that would cause a figure to be stretched horizontally by a factor of  $7/4$  and shrunk or squeezed vertically by a factor of  $2/3$ .

11. Write the inverse of the matrix from exercise (10) above. Verify that it is the inverse matrix.

12. Write a transformation matrix that would cause a figure to be slide 7 units to the right and then reflected over the x-axis.
13. Write the inverse of the matrix from exercise (12) above. Verify that it is the inverse matrix.
14. Write a transformation matrix that would cause a figure to be reflected over the y-axis and then slide 5 units down.
15. Matrix  $[F]$  below is a  $72^\circ$  CCW rotation matrix. Find the matrix product  $[F][F][F][F][F]$  or  $[F]^5$ .
- $$[F] = \begin{bmatrix} 0.3090 & -0.9511 & -2.4223 \\ 0.9511 & 0.3090 & -4.6660 \\ 0 & 0 & 1 \end{bmatrix}$$
16. Write a transformation matrix that would cause a figure to be rotated  $150^\circ$  counterclockwise about (0,0). Use the Sin and Cos keys on your graphing calculator. Round matrix elements to nearest 0.0001.  
**Warning!** Make sure that your calculator's angle mode is set to degree and **not** radian.
17. Write a transformation matrix that would cause a figure to be rotated  $100^\circ$  clockwise about (0,0). Use the Sin and Cos keys on your graphing calculator. Round matrix elements to nearest 0.0001.  
**Warning!** Make sure that your calculator's angle mode is set to degree and **not** radian.
18. Write a transformation matrix that would cause a figure to be rotated  $90^\circ$  counterclockwise about the point (-3,2).