Inverse Relations and Functions

(Every function is a relation, but not every relation is a function.)

Definition: The inverse of a relation is a relation obtained by reversing or swapping the coordinates of each ordered pair in the relation. If the relation is described by an equation in the variables **x** and **y**, the equation of the inverse relation is obtained by replacing every **x** in the equation with **y** and every **y** in the equation with **x**. Their graphs are mirror images over the line of reflection y = x. The graphs on the attached page are the graphs of the equations from examples 1, 2 and 3 given below. Study these graphs as you read the examples.

If f(x) represents a function of **x**, the inverse of the function is represented by the symbol $f^{-1}(x)$. **Warning!** $f^{-1}(x)$ does **NOT** equal 1 / f(x) . Simple example: If f(x) = 2x – 5, then $f^{-1}(x) = (x + 5) / 2$ and $f(x)^{-1} = 1 / (2x – 5)$.

Some Observations:

To find the domain and range of the inverse relation, swap the domain and range of the relation.

The squaring function and square root relation are inverse relations. ($y = x^2$ *and* $y = \pm\sqrt{x}$)

The cubing function and cube root function are inverse functions. ($y = x^3$ *and* $y = \sqrt[3]{x}$)

If a function is not one-to-one, the inverse of the function is a relation, but not a function.

The reciprocal function **y = 1/x** is its own inverse function.

The opposite function **y = - x** is its own inverse function.

The identity function **y = x** is its own inverse function.

$y = e^x$ and y = ln(x) are inverse functions. Example: $e^{2.5} = 12.18249396$ and ln(12.18249396) = 2.5 .

$y = 10^x$ and y = Log(x) are inverse functions. Example: $10^3 = 1{,}000$ and Log(1,000) = 3.

(Use the **MODE** key on your graphing calculator and set the angle mode to degree.)

y = sin(x) and $y = \sin^{-1}(x)$ are <u>almost</u> inverse relations. Example: Sin(45) = 0.7071067812 and $\text{Sin}^{-1}(0.7071067812) = 45$

Comment: In order for $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ to be mathematical functions, the range values of these functions are restricted. How the range values are restricted is explained in a trigonometry course.

Example 1: Let y = f(x) = 3/2x - 6. Find a formula for $f^{-1}(x)$ and show that the functions are inverse functions. **Note**: y = 3/2x – 6 is a one-to-one function and therefore its inverse will be a function.

Solution: Start with the equation **x = 3/2y - 6** and solve for y.

x = 3/2y - 6

x + 6 = 3/2y

2/3(x + 6) = y

2/3x + 4 = y

$f^{-1}(x) = 2/3x + 4$

Now show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$$f(2/3x+4) = \frac{3}{2}\left(\frac{2}{3}x+4\right) - 6 = x+6-6 = x$$

$$f^{-1}(3/2x-6) = \frac{2}{3}\left(\frac{3}{2}x-6\right) + 4 = x-4+4 = x$$

Comment: Notice that both f(x) and $f^{-1}(x)$ are linear and both have slopes that are reciprocals of each other (3/2 and 2/3) . You should be able to prove to yourself that inverse linear functions always have slopes that are reciprocals of each other . Also carefully study the graphs of f(x) and $f^{-1}(x)$ on the attached page.

Example 2: Let $f(x)=4\sqrt[3]{x+2}-1$ Find a formula for $f^{-1}(x)$ and show that the functions are inverse functions. **Note:** f(x) is a one-to-one function and therefore its inverse is a function.

Solution: Start with the equation $x=4\sqrt[3]{y+2}-1$ and solve for y.

$$x=4\sqrt[3]{y+2}-1$$

$$x+1=4\sqrt[3]{y+2}$$

$$\left(\frac{x+1}{4}\right)=\sqrt[3]{y+2}$$

$$\left(\frac{x+1}{4}\right)^3=y+2$$

$$\left(\frac{x+1}{4}\right)^3-2=y$$

$$f^{-1}(x)=\left(\frac{x+1}{4}\right)^3-2$$

Now show that f(f $^{-1}$(x)) = x and f $^{-1}$(f(x)) = x

$$f\left(\left(\frac{x+1)}{4}\right)^3-2\right)=4\sqrt[3]{\left(\frac{x+1}{4}\right)^3-2+2}-1=4\sqrt[3]{\left(\frac{x+1}{4}\right)^3}-1=$$

$$4\left(\frac{x+1}{4}\right)-1=x+1-1=x$$

$$f^{-1}\left(4\sqrt[3]{x+2}-1\right)=\left(\frac{4\sqrt[3]{x+2}-1+1}{4}\right)^3-2=\left(\sqrt[3]{x+2}\right)^3-2=x+2-2=x$$

Comment: Carefully study the graphs of f(x) and f $^{-1}$(x) on the next page.

Example 3: Let $f(x)=\frac{x+1}{x}$ Find a formula for f $^{-1}$(x) and show that the functions are inverse functions. f(x) is a one to-one function. Therefore its inverse is a function.

Solution: Start with the equation $x=\frac{y+1}{y}$ and solve for y.

$$x=\frac{y+1}{y}$$

xy = y + 1

xy - y = 1

y(x - 1) = 1

$$y=\frac{1}{x-1}$$

$$f^{-1}(x)=\frac{1}{x-1}$$

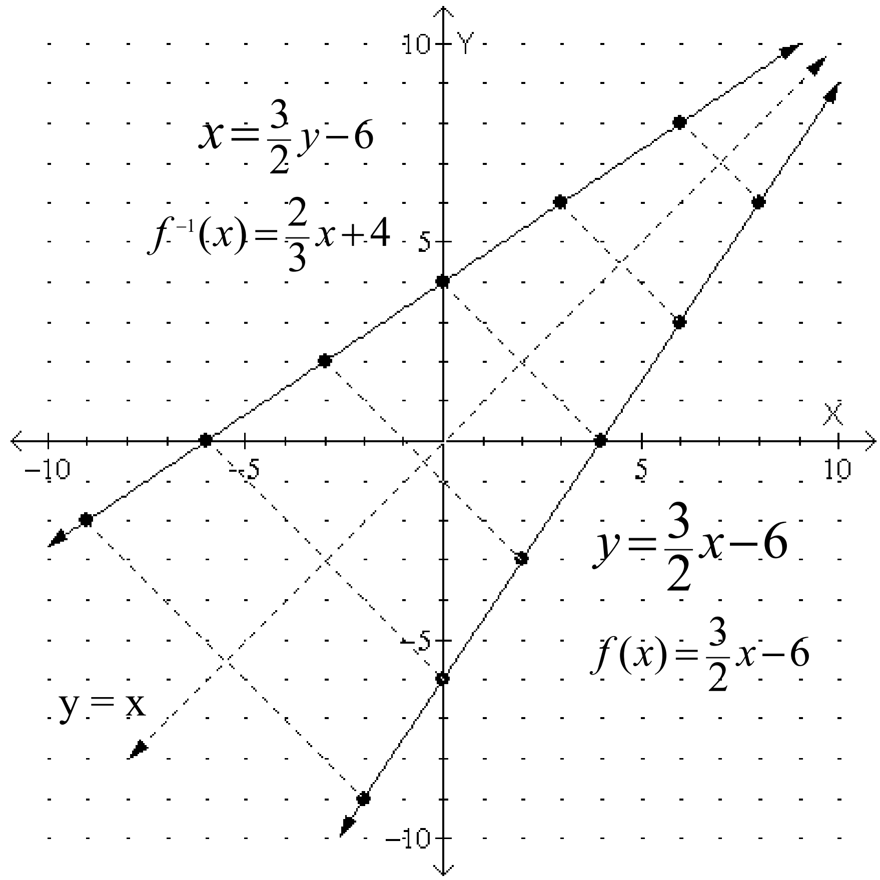
Now show that f(f $^{-1}$(x)) = x and f $^{-1}$(f(x)) = x

$$f\left(\frac{1}{x-1}\right)=\left(\frac{\frac{1}{x-1}+1}{\frac{1}{x-1}}\right)=\left(\frac{\frac{1+x-1}{x-1}}{\frac{1}{x-1}}\right)=\left(\frac{x}{x-1}\right)\left(\frac{x-1}{1}\right)=x$$

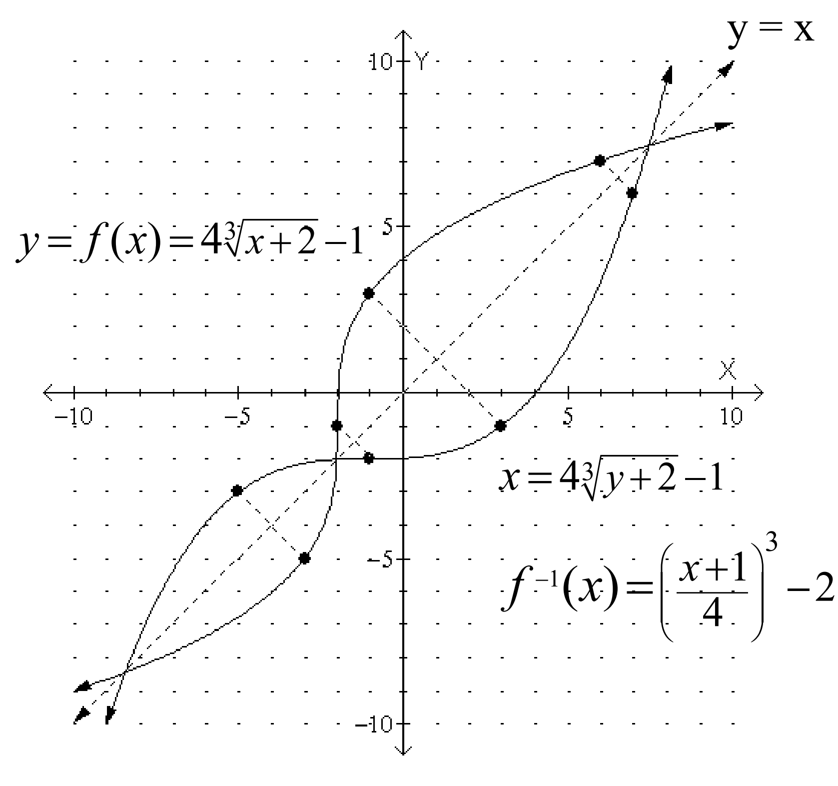
$$f^{-1}\left(\frac{x+1}{x}\right)=\left(\frac{1}{\frac{x+1}{x}-1}\right)=\left(\frac{1}{\frac{1}{x}}\right)=\left(\frac{1}{1}\right)\left(\frac{x}{1}\right)=x$$

Comment: Carefully study the graphs of f(x) and f $^{-1}$(x) on the next page.

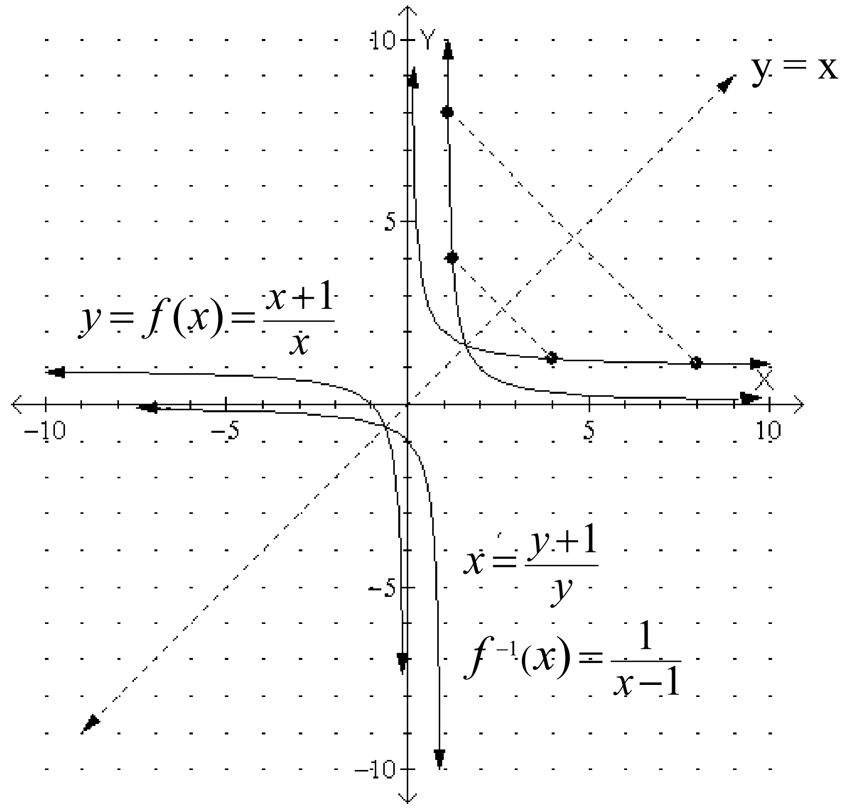
Example 1 Graphs



Example 2 Graphs



Example 3 Graphs



Comment: (y = f(x) = (x – 2)² + 3)

y = $(x-2)^2 + 3$ is a function, but not a one-to-one function. Therefore the inverse of f(x) is a relation, but not a function.

Equations for Inverse of f(x):

$$x = (y-2)^2 + 3$$

or

$$f^{-1}(x) = \pm\sqrt{x-3} + 2$$

Equations of Inverse Relation Circles

$$(x+6)^2 + (y-5)^2 = 9$$

$$(y+6)^2 + (x-5)^2 = 9$$