Matrices

Background: Matrices are rectangular grids of numbers arranged in rows and columns. Applications of matrices are found in many areas of mathematics. Areas of application include: 3D computer graphics, linear programming, Markov chains in probability theory, finite state automata theory, graph theory, network theory, multiple dimension vector spaces, economic modeling and solving systems of linear equations.

Order of a Matrix: The **order** of a matrix refers to the number of rows and columns of the matrix. A **n** x **m** matrix has **n** horizontal **rows** and **m** vertical **columns**. Most matrix calculations require matrices to have some type of specific order. If a matrix fails to have the required order, the matrix calculation can not be performed.

Examples:

a) A 1 x 5 matrix: **[2 0 -3 1000 -19]**

b) A 4 x 1 matrix: $\begin{bmatrix} 20.0000 \\ -3.0000 \\ 0.0000 \\ 217.0000 \end{bmatrix}$

c) A 3 x 6 matrix: $\begin{bmatrix} 2 & 0 & 1 & 0 & 10 & -1 \\ -1 & 0 & 0 & 4 & 7 & 10 \\ 12 & 5 & -12 & 18 & 3 & 0 \end{bmatrix}$

d) A 4 x 4 square matrix: $\begin{bmatrix} \cos(60) & 0 & -\sin(60) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(60) & 0 & \cos(60) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This matrix would be used to calculate the x-y-z coordinates of a figure in 3D space that has been rotated 60^{o} about the y – axis.

Symbols for Matrices: There are several methods to represent a matrix with symbols.

1) Use an uppercase letter to represent a matrix. The letters **A**, **B**, **C**, and **D** could be used to represent any of the four matrices shown above.

2) A representative element of the matrix enclosed in square brackets. Examples: $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.

3) Some computer programs require square brackets enclose uppercase letters. Examples: [**A**], [**B**], or [**C**].

4) Uppercase letter with a pair of subscripts to indicate the order of the matrix. Examples: $\mathbf{A}_{3\ 7}$ and $\mathbf{B}_{1\ 5}$.

Equality of Two Matrices: Two matrices are equal if and only if they have the same number of rows, the same number of columns, and all corresponding row-column elements are equal.

Example 1: The two matrices are equal.

$$\begin{bmatrix} 3 & 0 & -3 & 0.5 \\ -1 & 4 & 2 & -10 \\ -8 & 12 & 20 & -50 \end{bmatrix} = \begin{bmatrix} 6/2 & 1-1 & -3 & 1/2 \\ 2-3 & 4 & 10/5 & -10 \\ 2-10 & 2+10 & 20 & -60+10 \end{bmatrix}$$

Example 2: The two matrices are **not** equal. **Why?**

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

Scalar Times a Matrix: The operation of a scalar times a matrix is straightforward. Multiply each matrix row-column element by the scalar.

Example: If **A** = $\begin{bmatrix} 12 & 16 & -4 \\ 8 & -20 & 100 \\ 0 & 1 & -16 \\ 300 & -64 & 500 \end{bmatrix}$, then **-3/4 A** = $\begin{bmatrix} -9 & -12 & 3 \\ -6 & 15 & -75 \\ 0 & \frac{-3}{4} & 12 \\ -225 & 48 & -375 \end{bmatrix}$

Addition and Subtraction of Two Matrices: The operations of matrix addition and matrix subtraction are straightforward. If the two matrices have the same number of rows and the same number of columns, then they can be added or subtracted by adding or subtracting the corresponding matrix row-column elements.

Example 1: If $\mathbf{A} = \begin{bmatrix} 20 & 10 & 0 \\ 9 & -12 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 3 & 9 \\ -4 & -10 & 100 \end{bmatrix}$, then $2\mathbf{A} - \mathbf{B} = \begin{bmatrix} 39 & 17 & -9 \\ 22 & -14 & -90 \end{bmatrix}$

Multiplication of Two Matrices: The operation of multiplying two matrices is **NOT** straightforward. On first exposure to matrix multiplication, most people think it is very strange and way too complicated. However, with practice and experience with practical applications, matrix multiplication makes perfect sense. These are the matrix multiplication rules:

1. In order to multiply matrix **A** by matrix **B**, the number of columns in **A** must equal the number of rows in **B**. If this is not the case, the two matrices can **not** be multiplied and we say that they are **incompatible** for multiplication. If two matrices can be multiplied, we say that they are **compatible** for multiplication.

   Example: If **A** is a 7 x 3 matrix and **B** is a 3 x 2 matrix, then **A*B** is legal, but **B*A** is **NOT** possible!

2. If two matrices $\mathbf{A}_{n\,m}$ and $\mathbf{B}_{p\,q}$ are compatible for multiplication (m = p) as defined by rule (1) above, then the product of **A** and **B**, is a matrix with **n** rows and **q** columns.

   Example 1 : $A_{3\,1} * B_{1\,5}$ is 3 x 5 matrix and $A_{1\,3} * B_{3\,1}$ is a matrix with one element!

   Example 2: $A_{4\,4} * B_{4\,1}$ is 4 x 1 matrix and $B_{4\,1} * A_{4\,4}$ is an **impossible** matrix calculation!

3. If $\mathbf{A}_{m\,n} * \mathbf{B}_{p\,q} = \mathbf{C}_{m\,q}$, then n = p and the matrix element in row **j** and column **k** of matrix C equals the sum of the cross product elements of row **j** of matrix **A** and column **k** of matrix **B**.

   Example 1:
   $$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 5 & -1 \\ 10 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 28 \\ 7 \\ 20 \end{bmatrix}$$

   Example 2:
   $$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 5 & -1 \\ 10 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 10 & 1 \\ 9 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 64 & 62 & 10 \\ 1 & 46 & 5 \\ 20 & 40 & 35 \end{bmatrix}$$

   Example 3:
   $$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 10 & 1 \\ 9 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 5 & -1 \\ 10 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 14 & 7 \\ 14 & 63 & 2 \\ 18 & 56 & 50 \end{bmatrix}$$

   Example 4:
   $$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 5 & -1 & -9 \\ 10 & 5 & 0 & 49 \\ -6 & 21 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 5 & -1 & -9 \\ 10 & 5 & 0 & 49 \\ -6 & 21 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 5 & -1 & -9 \\ 10 & 5 & 0 & 49 \\ -6 & 21 & 0 & 5 \end{bmatrix}$$

   This example shows how the identity matrix works.

   Example 5:
   $$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The first two matrices on the left are inverse matrices of each other because the product of the matrices is the identity matrix.

Identity Matrix: An identity matrix is a square matrix with **1's** on the main diagonal and **zero's** everywhere else.

Examples: The 3 x 3 identity matrix = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the 4 x 4 identity matrix = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is **NOT** the 3 x 3 identity matrix! **Why is this so? See below.**

The identity matrix has the following properties:

* The identity matrix is represented by **I** .
* If **A** is any square matrix, then **AI = IA = A** . Look at matrix multiplication example 4 above.
* If **A** is compatible for multiplication with the identity matrix, then **AI = A or IA = A.**
* The identity matrix **I** works in the same manner that **1** works for multiplication of real numbers. Recall that **1** is the multiplicative identity for real numbers.

Inverse of a Square Matrix: Two square matrices **A** and **B** are inverses of each other if and only if **A*B = B*A** = **I** the identity matrix.

Example 1: $\begin{bmatrix} 11 & -12 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ -1 & \frac{11}{2} \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -1 & \frac{11}{2} \end{bmatrix} \begin{bmatrix} 11 & -12 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Look at matrix multiplication example 5 above.

Observations about the the **inverse** of a matrix:

* Only square matrices can have an inverse matrix and not all square matrices have inverses**.**
* A square matrix has an inverse matrix if and only if the row reduced echelon form of the matrix is the identity matrix.
* If matrix **A** is the inverse of matrix **B**, then **B** is the inverse of **A**.
* If matrix **A** and matrix **B** are inverses of each other, then **A*B** = **B*A** = **I** . This is one of the rare times that matrix multiplication commutes!
* If matrix **A** has an inverse matrix, then the inverse of **A** is denoted by $\mathbf{A^{-1}}$.
* $\mathbf{A^{-n}}$ means multiply the inverse of **A** by itself **n** times. Example: $\mathbf{A^{-3} = A^{-1} * A^{-1} * A^{-1}}$
* We do **NOT** divide matrices in the same sense that we divide real numbers. We multiply a matrix by the inverse of the matrix we want to divide by. Example: **A / B** means $\mathbf{A * B^{-1}}$.
* The inverse of a matrix is similar to the reciprocal or multiplicative inverse of a real number.
* When using matrices as geometric transformation functions (rotating, sliding, resizing, and flipping figures in 2D or 3D space), the inverse of a matrix is similar to the inverse of a real valued function f(x) . Recall that if f(x) is a real valued function and $f^{-1}(x)$ is its inverse, then f^{-1} **o** $f(x) = f^{-1}(f(\mathbf{x})) = \mathbf{I}(x) = \mathbf{x}$ and f **o** $f^{-1}(x) = f(f^{-1}(x)) = I(x) = \mathbf{x}$ where **I**(x) is the real valued identity function. Also $\mathbf{A * A^{-1} = A^{-1} * A = I}$ where I is the identity matrix.
* A matrix that **does not** have an inverse is called a **singular** matrix.

How to Find the Inverse of a Square Matrix: Using a graphing calculator or appropriate computer software, finding the inverse of a square matrix is a simple task. With only pencil and paper, finding the inverse of a square matrix is a daunting task. The following algorithm explains how to find the inverse of square matrix **A**.

1. Create a new matrix **B** so that the left-half of **B** = **A** and the right-half of **B** = the identity matrix **I** .
2. Transform the left-half of **B** into the identity matrix by doing elementary row operations on the left-half of **B.** These row operations will automatically change the right-half of matrix **B.**
3. When step (2) above is completed, the inverse of **A** equals the right-half of matrix **B .**

Example: Find $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 2 & 3 & \frac{1}{2} & 6 \\ 1 & 4 & -2 & -3 \\ 3 & 4 & -3 & -2 \\ -\frac{7}{10} & 6 & 2 & 1 \end{bmatrix}$

Form matrix $\mathbf{B} = \left[\begin{array}{cccc:cccc} 2 & 3 & \frac{1}{2} & 6 & 1 & 0 & 0 & 0 \\ 1 & 4 & -2 & -3 & 0 & 1 & 0 & 0 \\ 3 & 4 & -3 & -2 & 0 & 0 & 1 & 0 \\ \frac{-7}{10} & 6 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

Transform the left half of **B** into the identity matrix by doing elementary row operations on the left-half of **B**. Each row operation changes the right-half of **B .**

$$\mathbf{B} = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{-140}{331} & \frac{-1,510}{993} & \frac{1,210}{993} & \frac{410}{993} \\ 0 & 1 & 0 & 0 & \frac{25}{331} & \frac{82}{331} & \frac{-81}{662} & \frac{15}{331} \\ 0 & 0 & 1 & 0 & \frac{-176}{331} & \frac{-1,520}{993} & \frac{1,001}{993} & \frac{610}{993} \\ 0 & 0 & 0 & 1 & \frac{104}{331} & \frac{169}{331} & \frac{-142}{331} & \frac{-70}{331} \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{140}{331} & -\frac{1,510}{993} & \frac{1,210}{993} & \frac{410}{993} \\ \frac{25}{331} & \frac{82}{331} & -\frac{81}{662} & \frac{15}{331} \\ -\frac{176}{331} & -\frac{1,520}{993} & \frac{1,001}{993} & \frac{610}{993} \\ \frac{104}{331} & \frac{169}{331} & -\frac{142}{331} & -\frac{70}{331} \end{bmatrix}$$

Note: When using modern technology to solve problems involving matrix calculations, you will never see or need to see the results of most intermediate matrix calculations. You will not use the matrix inverse algorithm to calculate the inverse of a matrix. Just trust that these results are automatically calculated and stored in computer memory! **Truly amazing.**

Matrix Multiplication and Inverse of a Matrix to Solve a System of Linear Equations: One approach to solving a system of linear equations is to use the Gauss-Jordan algorithm to transform the system of equations to a row reduced echelon form matrix and then read the solutions off of the RREF matrix. Another method of solving a system of **n** linear equations with **n** variables is carried out as follows:

1. Rewrite the system of equations as a <u>matrix equation</u> of the form **[A][x] = [k]** where matrix **[A]** is an **n** x **n** matrix of equation coefficients, **[x]** is a **n x 1** matrix of equation variables, and **[k]** is a **n x 1** matrix of equation constants.

2. Enter matrix **[A]** and matrix **[k]** into your graphing calculator or favorite computer math program. Then calculate $[\mathbf{A}]^{-1}[\mathbf{k}]$. The solutions are the elements of the **n x 1** output matrix.

This is why it works.

$$[A][x] = [k] \qquad \text{// Original matrix equation}$$
$$[A]^{-1}[A][x] = [A]^{-1}[k] \qquad \text{// Multiply both sides of the equation by } [A]^{-1}$$
$$I[x] = [A]^{-1}[k] \qquad \text{// } [A]^{-1}[A] \text{ equals the identity matrix}$$
$$[x] = [A]^{-1}[k] \qquad \text{// The solution equals } [A]^{-1}[k] \text{ !!!}$$

Example: Solve the system of equations

$$\begin{aligned} 3x + 3y + 5z &= 1 \\ 3x + 5y + 9z &= 2 \\ 5x + 9y + 17z &= 4 \end{aligned}$$

1. Let matrix **[A]** = $\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}$, matrix **[x]** = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and matrix [k] = $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

2. Rewrite the system of equations as a <u>matrix equation</u>. $\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

3. Enter matrices **[A]** and **[k]** into your graphing calculator or favorite math program and compute

$$[A]^{-1}[k] = \begin{bmatrix} 0 \\ \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Therefore x = 0, y = - ½ and z = ½

Note: A <u>second approach</u> would be to let **[A]** = $\begin{bmatrix} 3 & 3 & 5 & 1 \\ 3 & 5 & 9 & 2 \\ 5 & 9 & 17 & 4 \end{bmatrix}$ and enter **[A]** into your graphing calculator. Then use your calculator's built in matrix **r**ow **r**educe **e**chelon **f**orm function **rref ([A])** to

This is the original matrix [A] with an additional 4th column of system constants.

compute rref(**[A]**) = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$

The solution appears in the last column of the output matrix. x = 0, y = - ½ , and z = ½ .