Equation Transformation Rules for Implicitly Defined Relations

Graph Transformation (Constant k > 0)	**Equation Transformation (Constant k > 0)**
Slide graph **k** units to the right in positive **x** direction.	Replace every **x** in the equation with **x – k** .
Slide graph **k** units to the left in negative **x** direction.	Replace every **x** in the equation with **x + k** .
Slide graph **k** units upward in positive **y** direction.	Replace every **y** in the equation with **y – k .**
Slide graph **k** units downward in negative **y** direction.	Replace every **y** in the equation with **y + k .**
Reflect the graph over the x-axis.	Replace every y in the equation with -y .
Reflect the graph over the y-axis.	Replace every x in the equation with -x .
Reflect the graph over the line y = x.	Replace every x with y and every y with x .

(Reflecting a graph over the line y = x, results in two graphs that are the graphs of inverse relations.)

Rotate graph 90^0 clockwise about (0,0).	Replace every x with -y and every y with x.
Rotate graph 90^0 counterclockwise about (0,0).	Replace every x with y and every y with -x.
Rotate graph 180^0 about (0,0).	Replace every x with -x and every y with -y.
Horizontally stretch or shrink graph by a factor of k.	Replace every x with x/k or (1/k)x.
Vertically stretch or shrink graph by a factor of k.	Replace every y with y/k or (1/k)y.

(If k > 1, the graph is stretched horizontally or vertically. If 0 < k < 1, the graph is shrunk horizontally or vertically.)

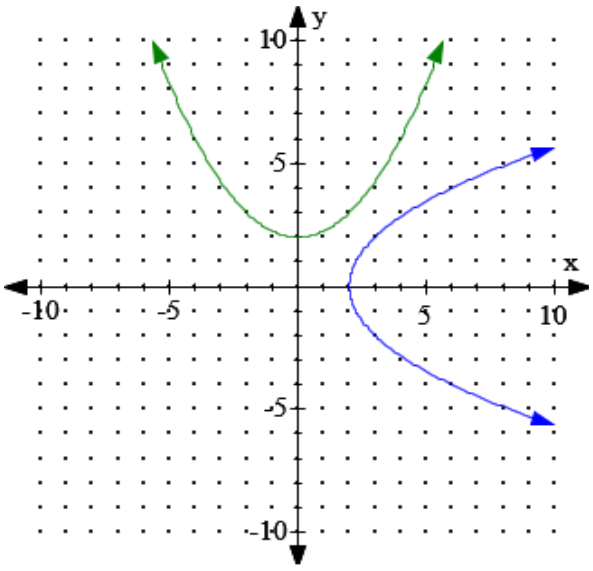
Reflect all graph points to the right the y-axis over the y-axis. All points to the right of the y-axis are fixed.	Replace every x with |x|.
Reflect all graph points above the x-axis over the x-axis. All points above of the x-axis are fixed.	Replace every y with |y|.

90^0 clockwise rotation of a graph about (0, 0):

Original equation $y = 0.25x^2 + 2$

Equation of rotated graph: $x = 0.25(-y)^2 + 2$

$x = 0.25y^2 + 2$

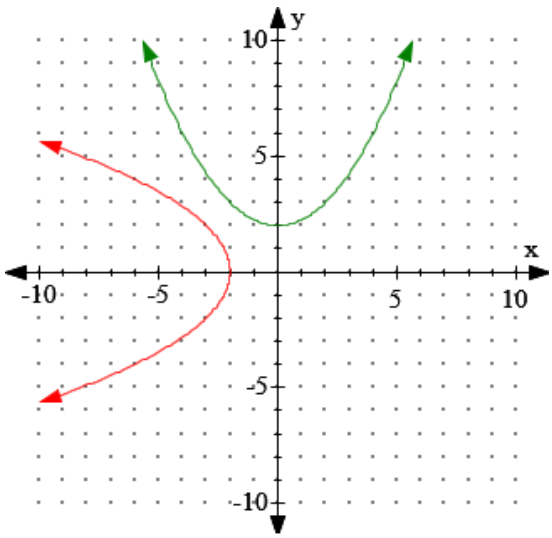


90^0 counterclockclockwise rotation of a graph about (0, 0)

Original equation $y = 0.25x^2 + 2$

Equation of rotated graph: $-x = 0.25y^2 + 2$

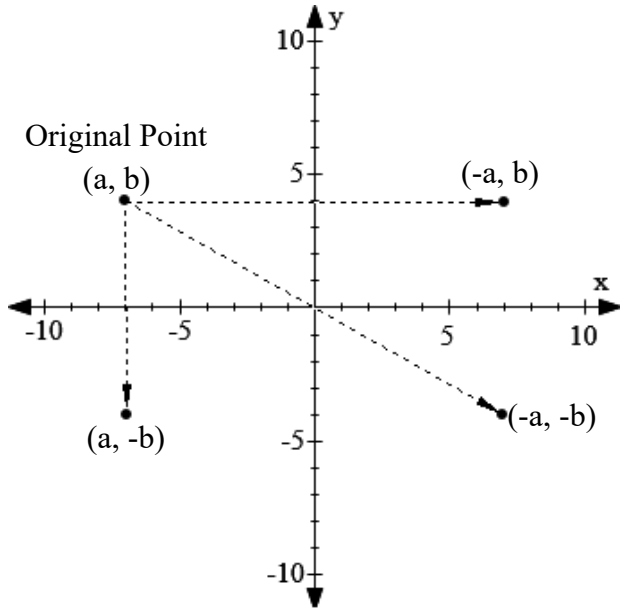
$x = -0.25y^2 - 2$



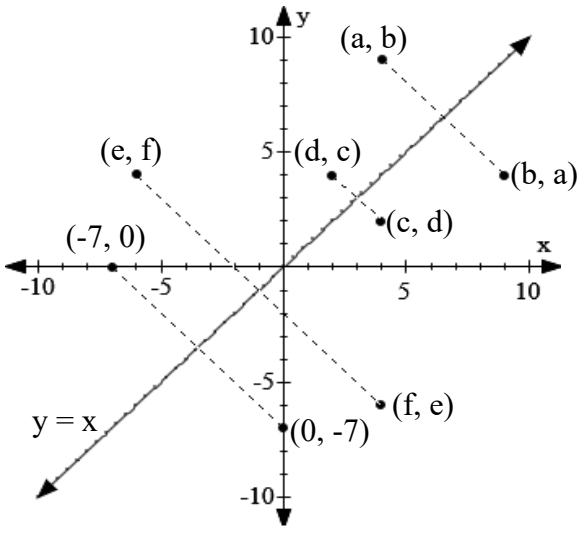
Equation Transformation Rules for Explicitly Defined Functions (y = f(x))

Graph Transformation (Constant k > 0)	**Equation Transformation (Constant k > 0)**
Slide graph **k** units to the right in positive **x** direction.	Replace every x with x – k. (y = f(x – k)).
Slide graph **k** units to the left in negative **x** direction.	Replace every x with x + k. (y = f(x + k))
Slide graph **k** units upward in positive **y** direction.	Replace every y with y – k. (y = f(x) + k)
Slide graph k units downward in negative y direction.	Replace every y with y + k. (y = f(x) – k))
Reflect the graph over the x-axis.	Replace every y with -y. (y = -f(x))
Reflect the graph over the y-axis.	Replace every x with -x. (y = f(-x))
Reflect the graph over the line y = x.	Replace all x with y and all y with x. (x = f(y))
(Reflecting a graph over the line y = x, results in two graphs that are the graphs of <u>inverse</u> relations.)	
Rotate graph 90^0 clockwise about (0,0).	Replace all x with -y and all y with x. (x = f(-y))
Rotate graph 90^0 counterclockwise about (0,0).	Replace all x with y and all y with -x. (x = -f(y))
Rotate graph 180^0 about (0,0).	Replace all x with -x and all y with -y. (y = -f(-x))
Horizontally stretch or shrink graph by a factor of k.	Replace all x with x/k or (1/k)x. (y = f(x/k))
Vertically stretch or shrink graph by a factor of k.	Replace all y with y/k or (1/k)y. (y = kf(x))
(If k > 1, the graph is stretched horizontally or vertically. If 0 < k < 1, the graph is shrunk horizontally or vertically.)	
Reflect all graph points to the right the y-axis over the y-axis. All points to the right of the y-axis are fixed.	Replace every x with |x|. (y = f(|x|))

Reflections: Over x-axis, over y-axis and about (0, 0). Refer to the transformation reflection rules and the 180^0 rotation transformation rule.



Reflection over the line y = x. Refer to the equation transformation rule about reflecting a graph over the line y = x. Notice that the line y = x is a perpendicular bisector of any line segment with endpoints at (x, y) and (y, x).



Why do the equation transformation rules work the way they do?

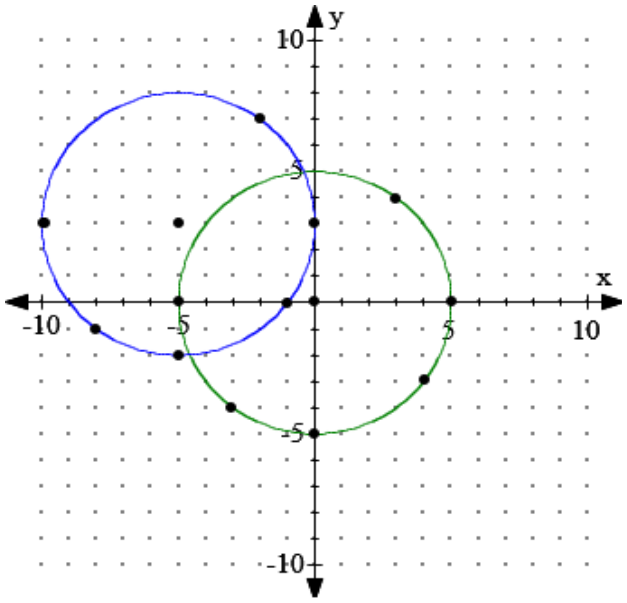
Many students find some of the equation transformation rules to be counterintutive. When we replace every instant of the variable x in an equation with x + 5, most students will guess that the graph is slid 5 units to the right in the positive direction, not 5 units left in the negative direction. When we replace every instance of the variable x in an equation with 2x, most students think that the graph will be stretched horizontally by a factor of 2, not shrunk horizontally by a factor of ½.

The two examples below show how the equation of a circle can be modified in order to achieve a desired transformation of the graph. After checking and comparing the solutions of both equations, it becomes clear why the transformation rules work.

Slide the graph 5 units to the left in the negative direction and up 3 units in the positive direction.

$x^2 + y^2 = 25$ $(x + 5)^2 + (y - 3)^2 = 25$

(3,4) ----------> (-2, 7)
(-4, 3) ----------> (-9, 6)
(0, -5) ----------> (-5, -2)
(5, 0) ----------> (0, 3)
(-3, -4) ----------> (-8, -1)
(4, -3) ----------> (-1, 0)
(-5, 0) ----------> (-10, 3)



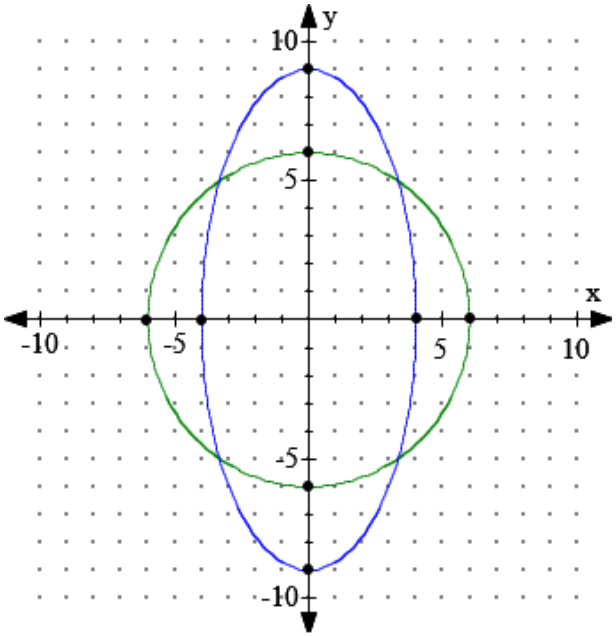
Shrink the graph horizontally by a factor of 2/3 and stretch the graph vertically by a factor of 1.5 = 3/2.

$x^2 + y^2 = 36$ $(3x/2)^2 + (2y/3)^2 = 36$

(6, 0) ----------> (4, 0)
(0, 6) ----------> (0, 9)
(-6, 0) ----------> (-4, 0)
(0, -6) ----------> (0, -9)

Note:
Horizontal shrink factor = 2/3, therefore replace every x with 3x/2.

Vertical stretch factor = 3/2, therefore replace every y with 2y/3.



Applying the Equation Transformation Rules

Most of the functions and relations that algebra through calculus students will encounter are the result of applying an ordered series of algebraic transformations to a basic equation. Each algebraic transformation that is applied to an equation results in a geometric transformation of the equation's graph. When one understands how the equation transformation rules work, it's a relatively easy task to find the equation of a graph given the graph of the equation, or to sketch the graph of an equation given the equation of the graph. A list of basic equations is shown below. Of course, which equations in the list that a student is required to understand (domain, range, function, relation, 1:1 function, and shape of graph) and memorize depends on the mathematical level of the student.

$y = x$ $y = \frac{k}{x}$ or $xy = k$ $y = |x|$ $y = x^2$ $y = \sqrt{x}$ $y = x^3$ $y = \sqrt[3]{x}$ $y = x^{2/3}$

$x^2 + y^2 = k^2$ $|x| + |y| = k$ $y = a \cdot b^{x/k}$ $y = e^x$ $y = \text{Log}(x)$ $y = \ln(x)$ $y = \text{floor}(x)$

$y = \text{Sin}(x)$ $y = \text{Cos}(x)$ $y = \text{Tan}(x)$ $y = \text{Csc}(x)$ $y = \text{Sec}(x)$ $y = \text{Cot}(x)$

Consider the basic square root function. A series of successive equation transformations will be applied to the basic equation in order to demonstrate the equation transformation rules.

Start with $y = \sqrt{x}$

Horizontally shrink the graph by a factor of $\frac{1}{2}$. ----> $y = \sqrt{2x}$

Vertically stretch the graph by a factor of 2. -----> $\frac{y}{2} = \sqrt{2x}$ or $y = 2\sqrt{2x}$

Reflect the graph over the y-axis. -----> $y = 2\sqrt{2(-x)}$ or $y = 2\sqrt{-2x}$

Reflect the graph over the x-axis. -----> $-y = 2\sqrt{-2x}$ or $y = -2\sqrt{-2x}$

Slide the graph horizontally 4 units in the negative direction. -----> $y = -2\sqrt{-2(x+4)}$ or $y = -2\sqrt{-2x-8}$